Multiscale Convergence Properties for Spectral Approximations of a Model Kinetic Equation

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Acknowledgements

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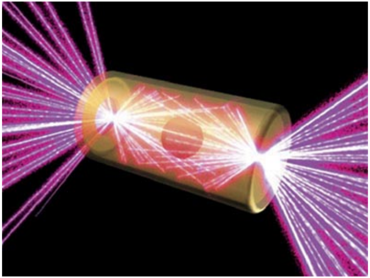
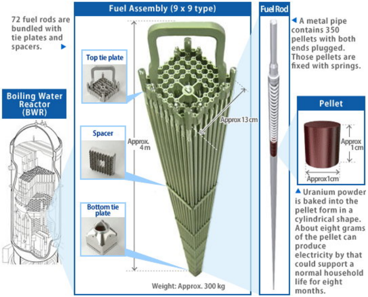
Outline

Introduction

Problem Setting



(a) Visualization of a Hohlraum



(b) Assembly for a Boiling Water Reactor

- Particles passing through a static material medium are **absorbed**, **scattered**, or **emitted** by the material.
- Particles are characterized by a **kinetic density** $f(x, E, \Omega, t)$ where
  - $x \in X \subset \mathbb{R}^3$ is the particle **position**
  - $E \in (0, \infty)$ is the particle **energy**
  - $\Omega \in \mathbb{S}^2$ is the particle **direction** of flight
  - $t \geq 0$ is **time**

Linear Transport Equation

- Cauchy problem for the **kinetic density** $f(x, \Omega, t)$:

$$\begin{cases} \partial_t f + \underbrace{\Omega \cdot \nabla_x f}_{\text{advection}} + \underbrace{\overbrace{\sigma_a f}^{\text{absorption}} + \overbrace{\sigma_s (f - \bar{f})}^{\text{scattering}}}_{\text{interactions}} = \underbrace{q}_{\text{source}} \\ f(x, \Omega, 0) = g(x, \Omega) \end{cases}$$

  - Mono-energetic particles with speed 1 (**single group**, so independent of E)
  - domain: $x \in \mathbb{R}^3$, $\Omega \in \mathbb{S}^2$, $t \in [0, \infty)$
  - cross-sections: σ_a and σ_s are absorption and scattering cross-sections
  - angular average: $\bar{f} = \frac{1}{4\pi} \int_{\mathbb{S}^2} f(x, \Omega, t) d\Omega$
  - source (emitting particles): q

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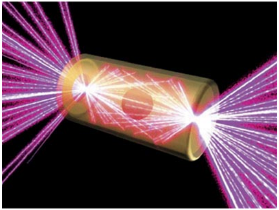
  - source (emitting particles): q

- Assumptions: static material, isotropic scattering, no absorption ($\sigma_a = 0$) and no source ($q = 0$)

$$\partial_t f + \Omega \cdot \nabla_x f + \sigma_s (f - \bar{f}) = 0$$

Applications

- Photons (fusion reactors, radiation in clouds, radiotherapy)
- Neutrinos (core-collapse supernova)
- Neutrons (fission reactors, imaging and scattering experiments)
- Electron transport in materials (semi-conductors, radiotherapy)
- Plasma simulations (fusion, plasma switches, space weather)



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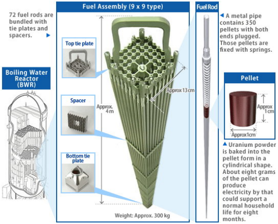


Assembly for a Boiling Water Reactor

Numerical Challenges

- High dimensional space: $x \in \mathbb{R}^3$, $\Omega \in \mathbb{S}^2$, 5D
- Coupling to the medium: nonlinearity, e.g. $\sigma_{\mathrm{s}}(f)$.
- **Multiscale dynamics**: diffusive, streaming, and transition regimes
  - Diffusive scaling
    - "strong scattering": $\sigma_{\mathrm{s}} \to \epsilon^{-1}\sigma_{\mathrm{s}}$
    - "long-time dynamics": $t \to \epsilon^{-1} t$
  - Scaled equation (WLOG, $\sigma_{\mathrm{s}} = 1$)

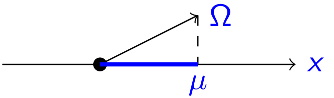
$$\epsilon \partial_t f + \Omega \cdot \nabla_x f + \frac{1}{\epsilon}(f - \bar{f}) = 0$$

Transport in 1-D

- Slab transport: the kinetic density $f(x, \mu, t)$ is the solution to

$$\epsilon \partial_t f + \mu \partial_x f + \frac{1}{\epsilon}(f - \bar{f}) = 0 \quad (1)$$

  - $x \in [-\pi, \pi)$: scalar coordinate along the axis perpendicular to a material slab; add periodic boundary condition
  - $\mu \in [-1, 1]$: the cosine of the angle between the x-axis and the direction of particle travel



  - $\bar{f} = \frac{1}{2} \int_{-1}^{1} f \, d\mu$: angular average
  - $\epsilon > 0$: a scaling parameter measuring the relative strength of different processes
  - initial condition $f(x, \mu, 0) = g(x, \mu)$

Transport in 1-D

- Slab transport:

$$\epsilon\partial_t f + \mu\partial_x f + \frac{1}{\epsilon}(f - \bar{f}) = 0 \tag{1}$$

- Denote the evolution operator as

$$\mathcal{T} \colon f \mapsto -\left(\mu\partial_x f + \frac{1}{\epsilon}(f - \bar{f})\right)$$

  then (1) becomes

$$\epsilon\partial_t f = \mathcal{T} f.$$

Spectral Approximation (Angular Discretization)

- Spectral approximation on μ: collision $\to$ smoothness
- Legendre expansion for f in μ:

$$f(x,\mu,t) = \sum_{\ell=0}^{\infty} f_\ell(x,t) p_\ell(\mu), \qquad f_\ell(x,t) = \int_{-1}^{1} p_\ell(\mu) f(x,\mu,t) d\mu,$$

where $p_\ell(\mu)$ are the normalized Legendre polynomials, which satisfy

$$\mu p_\ell = a_\ell p_{\ell+1} + a_{\ell-1} p_{\ell-1}.$$

- The coefficients/moments f_ℓ correspond to the physical quantities.

Spectral Approximation (Angular Discretization)

- slab transport:

$$\epsilon \partial_t f + \mu \partial_x f + \frac{1}{\epsilon}(f - \bar{f}) = 0$$

- Moment equations (infinite system):

$$\begin{cases} \epsilon \partial_t f_0 + a_0 \partial_x f_1 = 0, & \ell = 0; \\ \epsilon \partial_t f_\ell + a_\ell \partial_x f_{\ell+1} + a_{\ell-1} \partial_x f_{\ell-1} + \frac{1}{\epsilon} f_\ell = 0, & \ell \geq 1. \end{cases}$$

- P_N equations (closure of moments):

$$\begin{cases} \epsilon \partial_t f_0^N + a_0 \partial_x f_1^N = 0, & \ell = 0; \\ \epsilon \partial_t f_\ell^N + a_\ell \partial_x f_{\ell+1}^N + a_{\ell-1} \partial_x f_{\ell-1}^N + \frac{1}{\epsilon} f_\ell^N = 0, & 1 \leq \ell \leq N-1; \\ \epsilon \partial_t f_N^N + a_{N-1} \partial_x f_{N-1}^N + \frac{1}{\epsilon} f_N^N = 0, & \ell = N. \end{cases}$$

Motivation

Diffusion Limit

- Diffusion Approximation: in diffusive regimes (high scattering $\epsilon \ll 1$)

$$f = \rho + \mathcal{O}(\epsilon), \qquad \rho = \frac{1}{2}\int_{-1}^{1} f(x,\mu)\,d\mu$$

$$\partial_t \rho - \partial_x \left(\frac{1}{3} \partial_x \rho \right) = 0$$

- Formal results based on Hilbert expansion:
    - Larsen and Keller 1974
    - Habetler and Matkowsky 1975
- Rigorous analysis
    - Blankenship and Papanicolaou 1978
    - Bardos, Santos and Sentis 1984
- Cheap
- Inaccurate in non-diffusive regimes

Transition Regimes

- $0 < \epsilon < 1$, but not $\epsilon \ll 1$

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- Classical error estimate for spectral method:

$$||f(\cdot,\cdot,t) - f^N(\cdot,\cdot,t)||_{L^2(dxd\mu)} \leq \frac{C(t)}{N^q}.$$

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- Question: A good ϵ dependent estimate?

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- Question: A good ϵ dependent estimate?

  Guess: for $0 < \epsilon < 1$,

$$e^N = f - f^N = \mathcal{O}(\epsilon^{N+1}).$$

Errors of P_N methods

$$e^N := f - f^N = \eta + \xi$$

- Consistency error

$$\eta := f - \mathcal{P}(f) = \sum_{\ell=N+1}^{\infty} f_\ell(x,t) p_\ell(\mu)$$

- Stability error

$$\xi := \mathcal{P}(f) - f^N = \sum_{\ell=0}^{N} \xi_\ell(x,t) p_\ell(\mu)$$

with moment error

$$\xi_\ell = f_\ell - f_\ell^N.$$

Numerical Observation

Example: Consider

$$\begin{cases} \epsilon \partial_t f + \mu \partial_x f + \frac{1}{\epsilon} f = \frac{1}{\epsilon} \bar{f}, t \in (0,1], \\ f(0,x,\mu) = g^{(i)}(x,\mu), \end{cases}$$

for $[x,\mu] \in [-\pi,\pi) \times [-1,1]$, with periodic boundary condition and three different initial conditions:

$$g^{(1)} = (1+\mu-\mu^2)\left(1 + 1_{[-\frac{\pi}{2},\frac{\pi}{2}]}(x)\right),$$
$$g^{(2)} = (1+\mu-\mu^2)\left(1 + \cos x \, 1_{[-\frac{\pi}{2},\frac{\pi}{2}]}(x)\right),$$
$$g^{(3)} = (1+\mu-\mu^2)(1+\cos x).$$

$g^{(1)}$

ϵ	f_0^N	order	f_1^N	order	f_2^N	order	f_3^N	order	f_4^N	order
	0		1		2		3		4	
1/2	1.45E+00		1.01E-01		3.16E-02		1.36E-02		1.15E-02	
1/8	1.44E+00	0.00	2.22E-02	1.09	1.47E-03	2.21	1.10E-04	3.47	1.34E-05	4.87
1/32	1.44E+00	0.00	5.50E-03	1.01	9.06E-05	2.01	1.66E-06	3.02	4.74E-08	4.07
1/128	1.44E+00	0.00	1.37E-03	1.00	5.66E-06	2.00	2.59E-08	3.00	1.84E-10	4.00
1/512	1.44E+00	0.00	3.44E-04	1.00	3.54E-07	2.00	4.05E-10	3.00	7.18E-13	4.00

$g^{(2)}$

ϵ	f_0^N	order	f_1^N	order	f_2^N	order	f_3^N	order	f_4^N	order
1/2	1.27E+00		8.69E-02		2.83E-02		9.81E-03		5.35E-03	
1/8	1.27E+00	0.00	1.83E-02	1.12	1.34E-03	2.20	1.17E-04	3.20	1.27E-05	4.36
1/32	1.27E+00	0.00	4.52E-03	1.01	8.21E-05	2.01	1.77E-06	3.02	4.70E-08	4.04
1/128	1.27E+00	0.00	1.13E-03	1.00	5.13E-06	2.00	2.76E-08	3.00	1.83E-10	4.00
1/512	1.27E+00	0.00	2.82E-04	1.00	3.20E-07	2.00	4.31E-10	3.00	7.14E-13	4.00

$g^{(3)}$

ϵ	f_0^N	order	f_1^N	order	f_2^N	order	f_3^N	order	f_4^N	order
1/2	1.09E+00		1.53E-01		3.91E-02		1.15E-02		3.95E-03	
1/8	1.06E+00	0.02	3.48E-02	1.07	2.25E-03	2.06	1.43E-04	3.17	9.04E-06	4.39
1/32	1.06E+00	0.00	8.62E-03	1.01	1.39E-04	2.01	2.21E-06	3.01	3.47E-08	4.01
1/128	1.06E+00	0.00	2.15E-03	1.00	8.69E-06	2.00	3.44E-08	3.00	1.36E-10	4.00
1/512	1.06E+00	0.00	5.39E-04	1.00	5.43E-07	2.00	5.38E-10	3.00	5.30E-13	4.00

Table: Convergence rate of the coefficients f_ℓ^N in L^2 norm for P_4 method at $t = 1$.

$$f_\ell \sim f_\ell^N = \mathcal{O}(\epsilon^\ell), \qquad \ell = 0, 1, \ldots, N.$$

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ϵ	f_0^N	order	f_1^N	order	f_2^N	order	f_3^N	order	f_4^N	order
		0		1		2		3		4
1/2	1.45E+00		1.01E-01		3.16E-02		1.36E-02		1.15E-02	
1/8	1.44E+00	0.00	2.22E-02	1.09	1.47E-03	2.21	1.10E-04	3.47	1.34E-05	4.87
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1/2	1.27E+00		8.69E-02		2.83E-02		9.81E-03		5.35E-03	
1/8	1.27E+00	0.00	1.83E-02	1.12	1.34E-03	2.20	1.17E-04	3.20	1.27E-05	4.36
1/32	1.27E+00	0.00	4.52E-03	1.01	8.21E-05	2.01	1.77E-06	3.02	4.70E-08	4.04
1/128	1.27E+00	0.00	1.13E-03	1.00	5.13E-06	2.00	2.76E-08	3.00	1.83E-10	4.00
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Table: Convergence rate of the coefficients f_ℓ^N in L^2 norm for P_4 method at $t = 1$.

$$f_\ell \sim f_\ell^N = \mathcal{O}(\epsilon^\ell), \qquad \ell = 0, 1, \dots, N.$$

- $\eta = \mathcal{O}(\epsilon^{N+1})$. This is expected.

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ϵ	f_0^N	order 0	f_1^N	order 1	f_2^N	order 2	f_3^N	order 3	f_4^N	order 4
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1/8	1.27E+00	0.00	1.83E-02	1.12	1.34E-03	2.20	1.17E-04	3.20	1.27E-05	4.36
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1/128	1.06E+00	0.00	2.15E-03	1.00	8.69E-06	2.00	3.44E-08	3.00	1.36E-10	4.00
1/512	1.06E+00	0.00	5.39E-04	1.00	5.43E-07	2.00	5.38E-10	3.00	5.30E-13	4.00

Table: Convergence rate of the coefficients f_ℓ^N in L^2 norm for P_4 method at $t = 1$.

$$f_\ell \sim f_\ell^N = \mathcal{O}(\epsilon^\ell), \qquad \ell = 0, 1, \dots, N.$$

- $\eta = \mathcal{O}(\epsilon^{N+1})$. This is expected.
- What about ξ? And e^N?

$g^{(1)}$

ϵ	P_1 error	order $1+1=2$	P_2 error	order $2+1=3$	P_3 error	order $3+1=4$	P_4 error	order $4+1=5$	P_5 error	order $5+1=6$
1/2	5.62E-02		3.00E-02		1.86E-02		1.37E-02		1.09E-02	
1/8	2.05E-03	2.39	1.18E-04	4.00	1.35E-05	5.21	2.34E-06	6.26	4.42E-07	7.30
1/32	1.26E-04	2.01	1.67E-06	3.07	4.74E-08	4.08	2.00E-09	5.10	9.30E-11	6.11
1/128	7.88E-06	2.00	2.59E-08	3.00	1.84E-10	4.00	1.94E-12	5.01	2.25E-14	6.01
1/512	4.93E-07	2.00	4.05E-10	3.00	7.18E-13	4.00	1.89E-15	5.00	5.49E-18	6.00

$g^{(2)}$

ϵ	P_1 error	order	P_2 error	order	P_3 error	order	P_4 error	order	P_5 error	order
1/2	3.79E-02		1.32E-02		6.05E-03		3.28E-03		2.03E-03	
1/8	1.96E-03	2.14	1.20E-04	3.39	1.27E-05	4.45	1.54E-06	5.53	1.97E-07	6.66
1/32	1.21E-04	2.01	1.77E-06	3.04	4.70E-08	4.04	1.41E-09	5.05	4.44E-11	6.06
1/128	7.53E-06	2.00	2.76E-08	3.00	1.83E-10	4.00	1.37E-12	5.00	1.08E-14	6.00
1/512	4.70E-07	2.00	4.31E-10	3.00	7.14E-13	4.00	1.33E-15	5.00	2.63E-18	6.00

$g^{(3)}$

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1/2	4.70E-02		1.28E-02		4.01E-03		1.10E-03		2.50E-04	
1/8	2.79E-03	2.04	1.44E-04	3.24	9.04E-06	4.40	5.69E-07	5.46	3.57E-08	6.39
1/32	1.74E-04	2.00	2.21E-06	3.01	3.47E-08	4.01	5.46E-10	5.01	8.56E-12	6.01
1/128	1.08E-05	2.00	3.44E-08	3.00	1.36E-10	4.00	5.32E-13	5.00	2.09E-15	6.00
1/512	6.78E-07	2.00	5.38E-10	3.00	5.30E-13	4.00	5.20E-16	5.00	5.09E-19[1]	6.00

Table: Convergence rate of the L^2 error for P_1 to P_5 methods (compared with as a reference solution from P_{65}) at $t = 1$.

$$e^N = f - f^N = \mathcal{O}(\epsilon^{N+1})$$

[1]Multiprecision Computing Toolbox for MATLAB by Advanpix LLC. with 250 digits is used.

$g^{(1)}$

ϵ	ξ_0^N	order $2\times 4=8$	ξ_1^N	order $2\times 4+1=9$	ξ_2^N	order $2\times 4=8$	ξ_3^N	order $2\times 4-1=7$	ξ_4^N	order $2\times 4-2=6$
1/2	5.48E-03		3.36E-03		4.63E-03		4.02E-03		5.79E-03	
1/8	5.30E-08	8.33	1.08E-08	9.12	1.85E-08	8.96	8.13E-08	7.80	4.26E-07	6.86
1/32	7.16E-13	8.09	3.44E-14	9.13	2.14E-13	8.20	4.41E-12	7.09	9.30E-11	6.08
1/128	1.08E-17	8.01	1.30E-19	9.01	3.21E-18	8.01	2.67E-16	7.01	2.25E-14	6.01
1/512	1.65E-22	8.00	4.95E-25	9.00	4.89E-23	8.00	1.63E-20	7.00	5.50E-18	6.00

$g^{(2)}$

ϵ	ξ_0^N	order	ξ_1^N	order	ξ_2^N	order	ξ_3^N	order	ξ_4^N	order
1/2	7.92E-04		4.15E-04		7.04E-04		7.69E-04		1.24E-03	
1/8	1.76E-08	7.73	4.01E-09	8.33	5.46E-09	8.49	2.66E-08	7.41	1.93E-07	6.32
1/32	2.24E-13	8.13	1.28E-14	9.13	6.00E-14	8.24	1.50E-12	7.06	4.44E-11	6.05
1/128	3.38E-18	8.01	4.82E-20	9.01	8.97E-19	8.01	9.11E-17	7.00	1.08E-14	6.00
1/512	5.15E-23	8.00	1.84E-25	9.00	1.37E-23	8.00	5.56E-21	7.00	2.63E-18	6.00

$g^{(3)}$

ϵ	ξ_0^N	order	ξ_1^N	order	ξ_2^N	order	ξ_3^N	order	ξ_4^N	order
1/2	2.81E-07		1.11E-06		7.75E-06		4.68E-05		2.41E-04	
1/8	1.01E-10	5.72	2.26E-12	9.46	1.43E-10	7.86	2.25E-09	7.17	3.56E-08	6.36
1/32	1.59E-15	7.97	5.94E-18	9.27	2.14E-15	8.01	1.35E-13	7.01	8.57E-12	6.01
1/128	2.43E-20	8.00	2.20E-23	9.02	3.26E-20	8.00	8.23E-18	7.00	2.09E-15	6.00
1/512	3.71E-25	8.00	8.36E-29	9.00	4.97E-25	8.00	5.02E-22	7.00	5.10E-19	6.00

Table: Convergence rate of the L^2 error ξ_ℓ^N of the coefficients f_ℓ^N for P_4 method at $t=1$.

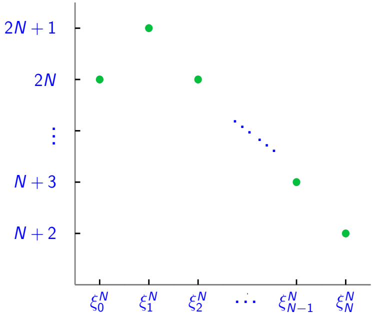
$$\xi_\ell^N := f_\ell - f_\ell^N = \begin{cases} \mathcal{O}(\epsilon^{2N}) & \ell = 0; \\ \mathcal{O}(\epsilon^{2N+2-\ell}), & \ell = 1,\dots,N. \end{cases}$$

- Formal $\xi_1^N = \mathcal{O}(\epsilon^{2N+1})$ [Larsen, Morel and McGhee 1996]
- Formal $\xi_\ell^N = \mathcal{O}(\epsilon^{2N+2-\ell})$ for $\ell \geq 1$ [Hauck and Lowrie 2009]

Observations

For all three initial conditions,

1. $e^N = \mathcal{O}(\epsilon^{N+1})$
2. a) $\xi_0^N = \mathcal{O}(\epsilon^{2N})$
   b) other $\xi_\ell^N = \mathcal{O}(\epsilon^{2N+2-\ell}) \implies$
3. $f_\ell^N = \mathcal{O}(\epsilon^\ell)$



Super convergence is important, because the low order moments correspond to the important physical quantities.

Theoretical results: isotropic i.c.

Theoretical results (isotropic i.c.)

Theorem (Multiscale convergence)

Given isotropic initial condition $g \in H^1(dx)$, *there exists an absolute constant* λ_1 *s.t.*

$$||f - f^N||_{L^2(dxd\mu)}(t) \leq B(g)e^{-\frac{\lambda_1 t}{\epsilon^2}} + C(\partial_x g)\sqrt{t}e^{-\frac{\lambda_1 t}{\epsilon^2}} + D(g, N, t)\epsilon^{N+1},$$

Moreover,

$$||f_\ell - f_\ell^N||_{L^2(dx)}(t) \leq \begin{cases} C(\partial_x g)\sqrt{t}e^{-\frac{\lambda_1 t}{\epsilon^2}} + E(g, N, 2, t)\epsilon^{2N}, & \ell = 0 \\ C(\partial_x g)\sqrt{t}e^{-\frac{\lambda_1 t}{\epsilon^2}} + E(g, N, \ell, t)\epsilon^{2N+2-\ell}, & 1 \leq \ell \leq N. \end{cases}$$

Notice, $\lambda_1 = \frac{1}{45}$; $D(g, N, t)$ and $E(g, N, \ell, t)$ are positive and bounded for any $t > 0$, and exponentially decreasing w.r.t t for t big enough. [2]

[2] Chen, Z., & Hauck, C. (2019). "Multiscale convergence properties for spectral approximations of a model kinetic equation." *Mathematics of Computation*, 88(319), 2257-2293.

Errors of P_N methods

$$e^N := f - f^N = \eta + \xi$$

- Consistency error

$$\eta := f - \mathcal{P}(f) = \sum_{\ell=N+1}^{\infty} f_\ell(x,t) p_\ell(\mu)$$

- Stability error

$$\xi := \mathcal{P}(f) - f^N = \sum_{\ell=0}^{N} \xi_\ell(x,t) p_\ell(\mu)$$

with moment error

$$\xi_\ell = f_\ell - f_\ell^N.$$

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The equation for ξ is

$$\epsilon \partial_t \xi = \mathcal{P}\mathcal{T}\xi - a_N p_N \partial_x f_{N+1}.$$

Asymptotic estimates on f_ℓ

Lemma

Given isotropic initial condition $g \in L^2(dx)$, the coefficients $f_\ell(t,x)$ of the solution f^N to the P_N equations satisfy

$$||f_\ell||_{L^2(dx)}(t) \leq B(g)e^{-\frac{\lambda_1 t}{\epsilon^2}} + C(g,\ell,t)\,\epsilon^\ell, \qquad \ell \geq 0.$$

Notice, $\lambda_1 = \frac{1}{45}$; $C(g,\ell,t)$ is bounded for any $t > 0$ and is monotonically decreasing w.r.t. t.

Some ideas

- Energy dissipation (periodic b.c.)

$$\partial_t \sum_{\ell=0}^{\infty} |f_\ell|^2 = -\frac{2}{\epsilon^2} \sum_{\ell=1}^{\infty} |f_\ell|^2 \leq 0$$

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- Formal analysis

$$\epsilon \partial_t f_\ell + a_\ell \partial_x f_{\ell+1} + a_{\ell-1} \partial_x f_{\ell-1} + \frac{1}{\epsilon} f_\ell = 0$$

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$$\partial_t \sum_{\ell=0}^{\infty} |f_\ell|^2 = -\frac{2}{\epsilon^2} \sum_{\ell=1}^{\infty} |f_\ell|^2 \leq 0$$

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$$\partial_t \sum_{\ell=0}^{\infty} |f_{\ell,k}|^2 = -\frac{2}{\epsilon^2} \sum_{\ell=1}^{\infty} |f_{\ell,k}|^2,$$

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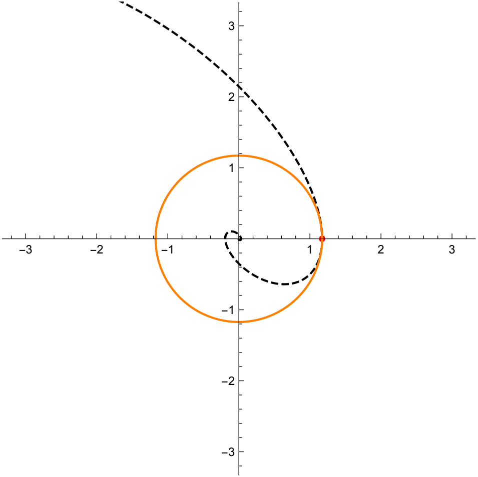
Separate frequencies to control the growth rates mode by mode, and then sum up over frequencies k.

Hypocoercivity

- Prof. David Levermore points out the hypocoercivity.
- [Dolbeault, Mouhot and Schmeiser 2015] the exponential rate of convergence to the equilibrium (unscaled linear kinetic equations)
- Example: P_1 with $k = 1$:



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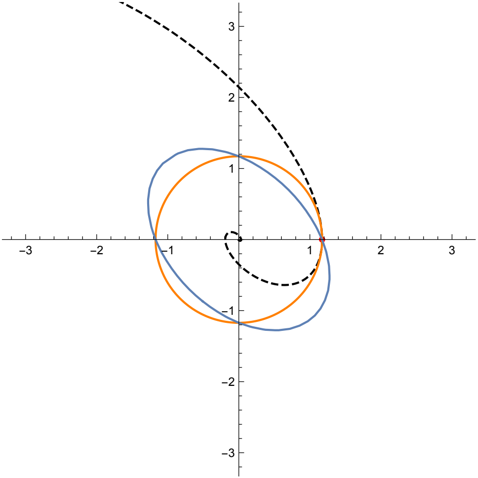

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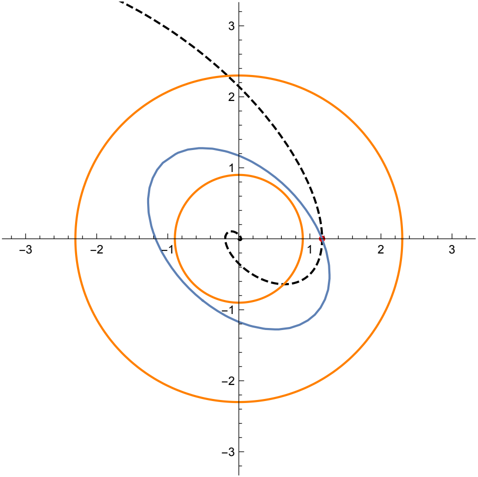


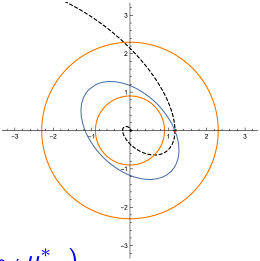
Hypocoercivity

- Rewrite energy dissipation as

$$\partial_t H_0^k(f) = -\frac{2}{\epsilon^2} H_1^k(f).$$

$$H_j^k(u) := \frac{1}{2}\sum_{\ell=j}^{\infty} |u_{\ell,k}|^2$$

Hypocoercivity



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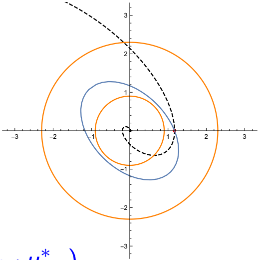
$$(H_0^k + h_\gamma^k)(u) \quad \text{and} \quad h_\gamma^k(u) = -\frac{\gamma}{4a_0}\,\mathrm{Im}(u_{0,k}\,u_{1,k}^*)$$

is a real-valued compensating function

Hypocoercivity



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  is a real-valued compensating function satisfying
  - equivalence

$$(1-\frac{\gamma}{2})H_0^k(u) \leq (H_0^k + h_\gamma^k)(u) \leq (1+\frac{\gamma}{2})H_0^k(u);$$

  - time derivative

$$\partial_t h_\gamma^k(u) \leq \gamma\left(\underbrace{-\frac{k}{16\epsilon}|u_{0,k}|^2}_{\text{"magic"}} + \underbrace{\left(\frac{k}{4\epsilon}+\frac{3}{8\epsilon^3 k}\right)|u_{1,k}|^2 + \frac{k}{5\epsilon}|u_{2,k}|^2}_{\text{"trade-off"}}\right).$$

Initial estimates on f_ℓ

- **Method of modified energy (MME)**: $(u \leftarrow f)$

$$\partial_t \left(H_0^k(f) + h_\gamma^k(f) \right) \leq -\frac{2}{\epsilon^2} H_3^k(f) - \sum_{\ell=0}^{2} c_{\gamma,\ell} |f_{\ell,k}|^2$$

where

$$c_{\gamma,0} = \frac{\gamma k}{16\epsilon}, \qquad c_{\gamma,1} = \frac{1}{\epsilon^2} - \frac{\gamma k}{4\epsilon} - \frac{3\gamma}{8\epsilon^3 k}, \qquad c_{\gamma,2} = \frac{1}{\epsilon^2} - \frac{\gamma k}{5\epsilon}.$$

For each k, choose suitable γ, and find positive lower bounds on $\{c_{\gamma,i}\}_{i=0}^2$.

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(i) High frequency: $(k\epsilon > 1/2)$

$$\gamma := \gamma^{\text{high}} \simeq \frac{1}{k\epsilon}$$

$$H_0^k(f)(t) < 6 H_0^k(g)\, e^{-\frac{2\lambda_1 t}{\epsilon^2}}$$

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(ii) Low frequency: $(0 < k\epsilon \le \frac{1}{2})$

$$\gamma := \gamma^{\text{low}} \simeq k\epsilon$$

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$$H_0^k(f) := \frac{1}{2}\sum_{\ell=0}^{\infty} |f_{\ell,k}|^2$$

$\Downarrow$

$f_{\ell,k}$

$\Downarrow$ sum over k

f_ℓ

Initial estimates on f_ℓ

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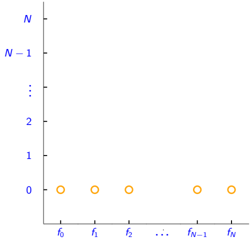

Initial estimate:

$$\|f_\ell\|^2_{L^2(dx)}(t) \leq 12 \overbrace{\sum_{|k|\epsilon > 1/2} H_0^k(g) e^{-\frac{2\lambda_1 t}{\epsilon^2}}}^{\text{high frequencies}} + \underbrace{C(g,0,t)^2}_{\text{low frequencies}}$$

with

$$C(g,0,t) = \left[24 \max_{k>0} H_0^k(g) \sum_{k>0} e^{-2\lambda_2 k^2 t}\right]^{\frac{1}{2}}.$$

Initial estimates on f_ℓ

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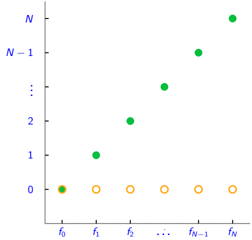


- **Next goal:** f_ℓ is bounded by **a finer estimate**:

$$\|f_\ell\|^2_{L^2(dx)}(t) \le \overbrace{12 \sum_{|k|\epsilon > 1/2} H_0^k(g) e^{-\frac{2\lambda_1 t}{\epsilon^2}}}^{\text{high frequencies}} + \underbrace{C(g, \ell, t)^2 \epsilon^{2\ell}}_{\text{low frequencies}}$$

with

$$C(g, \ell, t) = \left[24 \max_{k>0} H_0^k(g) \sum_{k>0} (Ak)^{2\ell} e^{-2\lambda_2 k^2 t} \right]^{\frac{1}{2}}.$$

Zoom-in estimates on f_ℓ

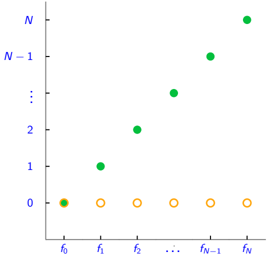
- Moment equations

$$\begin{cases} \epsilon\partial_t f_{0,k} + a_0 ikf_{1,k} = 0, & \ell = 0; \\ \epsilon\partial_t f_{\ell,k} + a_\ell ikf_{\ell+1,k} + a_{\ell-1} ikf_{\ell-1,k} + \frac{1}{\epsilon} f_{\ell,k} = 0, & \ell \geq 1. \end{cases}$$

For low frequency terms ($0 < |k|\epsilon \leq \frac{1}{2}$): "Zoom-in procedure"

---

(0) For $n \geq 0$, $|f_{n,k}|(t) \leq C_0^k \epsilon^0 k^0 e^{-\lambda_2 k^2 t}$.



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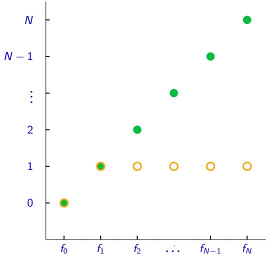
(1) For $n \geq 1$,

$$\partial_t |f_{n,k}| + \frac{1}{\epsilon^2}|f_{n,k}| \leq \frac{k}{\epsilon}\left(a_n|f_{n+1,k}| + a_{n-1}|f_{n-1,k}|\right)$$
$$\leq \frac{2k}{\sqrt{3}\epsilon}\left(C_0^k \epsilon^0 k^0 e^{-\lambda_2 k^2 t}\right).$$

$$|f_{n,k}|(t) \leq e^{-\frac{t}{\epsilon^2}} g_{n,k} \nearrow 0 + \frac{2k}{\sqrt{3}\epsilon}\int_0^t e^{-\frac{t-s}{\epsilon^2}}\left(C_0^k \epsilon^0 k^0 e^{-\lambda_2 k^2 s}\right) ds$$
$$\leq C_{0+1}^k \epsilon^{0+1} k^{0+1} e^{-\lambda_2 k^2 t}$$



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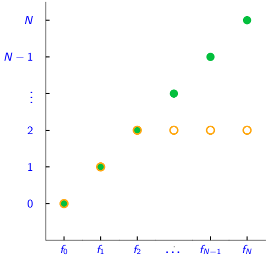
(1) For $n \geq 1$, $|f_{n,k}|(t) \leq C_1^k \epsilon^1 k^1 e^{-\lambda_2 k^2 t}$.

(2) For $n \geq 2$,

$$\begin{aligned} \partial_t |f_{n,k}| + \frac{1}{\epsilon^2}|f_{n,k}| &\leq \frac{k}{\epsilon}\left(a_n|f_{n+1,k}| + a_{n-1}|f_{n-1,k}|\right) \\ &\leq \frac{2k}{\sqrt{3}\epsilon}\left(C_1^k \epsilon^1 k^1 e^{-\lambda_2 k^2 t}\right). \end{aligned}$$

$$\begin{aligned} |f_{n,k}|(t) &\leq e^{-\frac{t}{\epsilon^2}} \cancelto{0}{g_{n,k}} + \frac{2k}{\sqrt{3}\epsilon}\int_0^t e^{-\frac{t-s}{\epsilon^2}}\left(C_1^k \epsilon^1 k^1 e^{-\lambda_2 k^2 s}\right) ds \\ &\leq C_{1+1}^k \epsilon^{1+1} k^{1+1} e^{-\lambda_2 k^2 t} \end{aligned}$$



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(0) For $n \geq 0$, $|f_{n,k}|(t) \leq C_0^k \epsilon^0 k^0 e^{-\lambda_2 k^2 t}$.

(1) For $n \geq 1$, $|f_{n,k}|(t) \leq C_1^k \epsilon^1 k^1 e^{-\lambda_2 k^2 t}$.

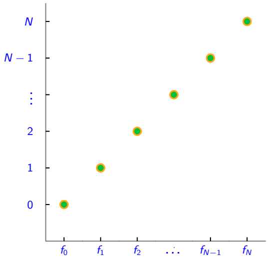
(2) For $n \geq 2$, $|f_{n,k}|(t) \leq C_2^k \epsilon^2 k^2 e^{-\lambda_2 k^2 t}$.

$\Downarrow$ Method of induction

(ℓ) For $n \geq \ell$, $|f_{n,k}|(t) \leq C_\ell^k \epsilon^\ell k^\ell e^{-\lambda_2 k^2 t}$.

$\Downarrow$ Take finest ϵ-estimate

For any $\ell \geq 0$, $|f_{\ell,k}|(t) \leq C_\ell^k \epsilon^\ell \underbrace{k^\ell e^{-\lambda_2 k^2 t}}_{\text{summable over k}}$.



Zoom-in estimates

- Moment equations

$$\begin{cases} \epsilon\partial_t f_{0,k} + a_0 ikf_{1,k} = 0, & \ell = 0; \\ \epsilon\partial_t f_{\ell,k} + a_\ell ikf_{\ell+1,k} + a_{\ell-1} ikf_{\ell-1,k} + \frac{1}{\epsilon} f_{\ell,k} = 0, & \ell \geq 1. \end{cases}$$

1. $k = 0$ terms: $f_{0,0}(t) = f_{0,0}(0)$, $\quad \epsilon\partial_t f_{\ell,0} + \frac{1}{\epsilon} f_{\ell,0} = 0$ for $\ell \geq 1$
2. High frequency terms, $|k|\epsilon > \frac{1}{2}$: decay exponentially $\mathcal{O}(e^{-\frac{\lambda_1 t}{\epsilon^2}})$
3. Low frequency terms, $0 < |k|\epsilon \leq \frac{1}{2}$: $f_{\ell,k}$ behaves like $\mathcal{O}(\epsilon^\ell)$

Lemma (Asymptotic approximation)

Given isotropic initial condition $g \in L^2(dx)$, the coefficients $f_\ell(t,x)$ of the solution f^N to the P_N equations satisfy

$$||f_\ell||_{L^2(dx)}(t) \leq B(g) e^{-\frac{\lambda_1 t}{\epsilon^2}} + C(g,\ell,t)\epsilon^\ell, \qquad \ell \geq 0.$$

Consistency error

$$||\eta||^2_{L^2(d\mu dx)}(t) = \sum_{\ell=N+1}^{\infty} ||f_\ell||^2_{L^2(dx)}(t)$$

Lemma

Given isotropic initial condition $g \in L^2(dx)$,

$$||\eta||_{L^2(d\mu dx)}(t) \leq B(g)e^{-\frac{\lambda_1 t}{\epsilon^2}} + \sqrt{3}C(g, N+1, t)\epsilon^{N+1}.$$

Stability error

- Stability error equation:

$$\begin{cases} \epsilon \partial_t \xi_{0,k} + a_0 ik \xi_{1,k} = 0, & \ell = 0; \\ \epsilon \partial_t \xi_{\ell,k} + a_\ell ik \xi_{\ell+1,k} + a_{\ell-1} ik \xi_{\ell-1,k} + \frac{1}{\epsilon} \xi_{\ell,k} = 0, & 1 \le \ell \le N-1; \\ \epsilon \partial_t \xi_{N,k} + a_{N-1} ik \xi_{N-1,k} + \frac{1}{\epsilon} \xi_{N,k} = -a_N ikf_{N+1,k}, & \ell = N. \end{cases}$$

Stability error

- Stability error equation:

$$\begin{cases} \epsilon \partial_t \xi_{0,k} + a_0 ik \xi_{1,k} = 0, & \ell = 0; \\ \epsilon \partial_t \xi_{\ell,k} + a_\ell ik \xi_{\ell+1,k} + a_{\ell-1} ik \xi_{\ell-1,k} + \frac{1}{\epsilon} \xi_{\ell,k} = 0, & 1 \le \ell \le N-1; \\ \epsilon \partial_t \xi_{N,k} + a_{N-1} ik \xi_{N-1,k} + \frac{1}{\epsilon} \xi_{N,k} = -a_N ikf_{N+1,k}, & \ell = N. \end{cases}$$

$$\partial_t \left(H_0^k(\xi) + h_\gamma^k(\xi) \right) \le -\frac{1}{\epsilon^2} H_3^k(\xi) - \sum_{\ell=0}^{2} c_{\gamma,\ell} |\xi_{\ell,k}|^2 + \frac{k^2}{6} |f_{N+1,k}|^2$$

- **Remark 1:**

$$\partial_t \left(H_0^k(f) + h_\gamma^k(f) \right) \le -\frac{2}{\epsilon^2} H_3^k(f) - \sum_{\ell=0}^{2} c_{\gamma,\ell} |f_{\ell,k}|^2$$

Stability error

- Stability error equation:

$$\begin{cases} \epsilon\partial_t \xi_{0,k} + a_0 ik\xi_{1,k} = 0, & \ell = 0; \\ \epsilon\partial_t \xi_{\ell,k} + a_\ell ik\xi_{\ell+1,k} + a_{\ell-1} ik\xi_{\ell-1,k} + \frac{1}{\epsilon}\xi_{\ell,k} = 0, & 1 \le \ell \le N-1; \\ \epsilon\partial_t \xi_{N,k} + a_{N-1} ik\xi_{N-1,k} + \frac{1}{\epsilon}\xi_{N,k} = -a_N ikf_{N+1,k}, & \ell = N. \end{cases}$$

$$\partial_t \left(H_0^k(\xi) + h_\gamma^k(\xi) \right) \le -\frac{1}{\epsilon^2} H_3^k(\xi) - \sum_{\ell=0}^{2} c_{\gamma,\ell} |\xi_{\ell,k}|^2 + \frac{k^2}{6} |f_{N+1,k}|^2$$

- **Remark 1:**

$$\partial_t \left(H_0^k(f) + h_\gamma^k(f) \right) \le -\frac{2}{\epsilon^2} H_3^k(f) - \sum_{\ell=0}^{2} c_{\gamma,\ell} |f_{\ell,k}|^2$$

- **Remark 2:**

$$\xi(0) = 0$$

- **Remark 3:**

$$||f_{N+1}|| = \mathcal{O}(\epsilon^{N+1})$$

Stability error

- Stability error equation:

$$\begin{cases} \epsilon\partial_t\xi_{0,k} + a_0 ik\xi_{1,k} = 0, & \ell = 0; \\ \epsilon\partial_t\xi_{\ell,k} + a_\ell ik\xi_{\ell+1,k} + a_{\ell-1} ik\xi_{\ell-1,k} + \frac{1}{\epsilon}\xi_{\ell,k} = 0, & 1 \le \ell \le N-1; \\ \epsilon\partial_t\xi_{N,k} + a_{N-1} ik\xi_{N-1,k} + \frac{1}{\epsilon}\xi_{N,k} = -a_N ikf_{N+1,k}, & \ell = N. \end{cases}$$

$$\partial_t \left(H_0^k(\xi) + h_\gamma^k(\xi) \right) \le -\frac{1}{\epsilon^2} H_3^k(\xi) - \sum_{\ell=0}^{2} c_{\gamma,\ell} |\xi_{\ell,k}|^2 + \frac{k^2}{6} |f_{N+1,k}|^2,$$

(i) **High frequency:** With estimate on source of the error coefficient system

$$|f_{N+1,k}|^2(t) \le 12 H_0^k(g)\, e^{-\frac{2\lambda_1 t}{\epsilon^2}}$$

$$\xrightarrow{\text{MME}} \quad H_0^k(\xi)(t) \le 6t\, k^2 H_0^k(g)\, e^{-\frac{2\lambda_1 t}{\epsilon^2}} = 6t\, H_0^k(\partial_x g)\, e^{-\frac{2\lambda_1 t}{\epsilon^2}}.$$

Stability error

- Stability error equation:

$$\begin{cases} \epsilon\partial_t\xi_{0,k} + a_0 ik\xi_{1,k} = 0, & \ell = 0; \\ \epsilon\partial_t\xi_{\ell,k} + a_\ell ik\xi_{\ell+1,k} + a_{\ell-1} ik\xi_{\ell-1,k} + \frac{1}{\epsilon}\xi_{\ell,k} = 0, & 1 \le \ell \le N-1; \\ \epsilon\partial_t\xi_{N,k} + a_{N-1} ik\xi_{N-1,k} + \frac{1}{\epsilon}\xi_{N,k} = -a_N ikf_{N+1,k}, & \ell = N. \end{cases}$$

$$\partial_t\left(H_0^k(\xi) + h_\gamma^k(\xi)\right) \le -\frac{1}{\epsilon^2}H_3^k(\xi) - \sum_{\ell=0}^{2} c_{\gamma,\ell}|\xi_{\ell,k}|^2 + \frac{k^2}{6}|f_{N+1,k}|^2,$$

(ii) Low frequency: With estimate on source of the error coefficient system

$$|f_{N+1,k}|(t) \le C_{N+1}^k \epsilon^{N+1} k^{N+1} e^{-\lambda_2 k^2 t}$$

$$\xrightarrow{\text{MME}} \quad H_0^k(\xi)(t) \le \frac{t}{2}(C_{N+1}^k)^2 k^{2(N+2)} e^{-2\lambda_2 k^2 t}\, \epsilon^{2(N+1)}.$$

Stability error

- Stability error equation:

$$\begin{cases} \epsilon\partial_t\xi_{0,k} + a_0 ik\xi_{1,k} = 0, & \ell = 0; \\ \epsilon\partial_t\xi_{\ell,k} + a_\ell ik\xi_{\ell+1,k} + a_{\ell-1} ik\xi_{\ell-1,k} + \frac{1}{\epsilon}\xi_{\ell,k} = 0, & 1 \le \ell \le N-1; \\ \epsilon\partial_t\xi_{N,k} + a_{N-1} ik\xi_{N-1,k} + \frac{1}{\epsilon}\xi_{N,k} = -a_N ikf_{N+1,k}, & \ell = N. \end{cases}$$

$$\partial_t\left(H_0^k(\xi) + h_\gamma^k(\xi)\right) \le -\frac{1}{\epsilon^2}H_3^k(\xi) - \sum_{\ell=0}^{2} c_{\gamma,\ell}|\xi_{\ell,k}|^2 + \frac{k^2}{6}|f_{N+1,k}|^2,$$

(i) High frequency: $H_0^k(\xi)(t) \le 6t\, H_0^k(\partial_x g)\, e^{-\frac{2\lambda_1 t}{\epsilon^2}}$

(ii) Low frequency: $H_0^k(\xi)(t) \le \frac{t}{2}[C_{N+1}^k]^2 k^{2(N+2)} e^{-2\lambda_2 k^2 t}\, \epsilon^{2(N+1)}$

Lemma

Given isotropic initial condition $g \in H^1(dx)$,

$$||\xi||_{L^2(d\mu dx)}(t) \le C(\partial_x g)\sqrt{t}e^{-\frac{\lambda_1 t}{\epsilon^2}} + \frac{\sqrt{t}}{A}C(g, N+2, t)\epsilon^{N+1}.$$

P_N error

$$||\eta||_{L^2(d\mu dx)}(t) \leq B(g)e^{-\frac{\lambda_1 t}{\epsilon^2}} + \sqrt{3}C(g, N+1, t)\epsilon^{N+1}.$$

$$+$$

$$||\xi||_{L^2(d\mu dx)}(t) \leq C(\partial_x g)\sqrt{t}e^{-\frac{\lambda_1 t}{\epsilon^2}} + \frac{\sqrt{t}}{A}C(g, N+2, t)\epsilon^{N+1}.$$

$$\Downarrow$$

$$||e^N||_{L^2(dxd\mu)}(t) \leq \left(B(g) + C(\partial_x g)\sqrt{t}\right) e^{-\frac{\lambda_1 t}{\epsilon^2}} + D(g, N, t)\epsilon^{N+1},$$

P_N error

$$||\eta||_{L^2(d\mu dx)}(t) \leq B(g)e^{-\frac{\lambda_1 t}{\epsilon^2}} + \sqrt{3}C(g, N+1, t)\epsilon^{N+1}.$$

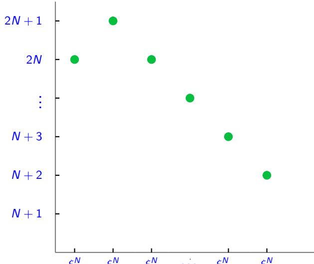
$$+$$

$$||\xi||_{L^2(d\mu dx)}(t) \leq C(\partial_x g)\sqrt{t}e^{-\frac{\lambda_1 t}{\epsilon^2}} + \frac{\sqrt{t}}{A}C(g, N+2, t)\epsilon^{N+1}.$$

$$\Downarrow$$

$$||e^N||_{L^2(dxd\mu)}(t) \leq \left(B(g) + C(\partial_x g)\sqrt{t}\right) e^{-\frac{\lambda_1 t}{\epsilon^2}} + D(g, N, t)\epsilon^{N+1},$$

Next: moment errors $\xi_\ell = f_\ell - f_\ell^N$
with more tricky "Zoom-in" techniques



Moment errors

Error equations:

$$\begin{cases} \epsilon\partial_t \xi_{0,k} + a_0 ik\xi_{1,k} = 0, & \ell = 0; \\ \epsilon\partial_t \xi_{\ell,k} + a_l ik\xi_{\ell+1,k} + a_{\ell-1} ik\xi_{\ell-1,k} + \frac{1}{\epsilon}\xi_{\ell,k} = 0, & 1 \le \ell \le N-1; \\ \epsilon\partial_t \xi_{N,k} + a_{N-1} ik\xi_{N-1,k} + \frac{1}{\epsilon}\xi_{N,k} = -a_N ikf_{N+1,k}, & \ell = N. \end{cases}$$

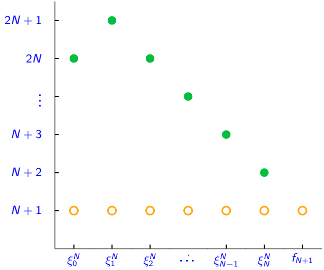
For low frequency terms ($0 < |k|\epsilon \le \frac{1}{2}$): "Zoom-in procedure"

The previous result

$$H_0^k(\xi)(t) \le \frac{t}{2}[C_{N+1}^k]^2 k^{2(N+2)} e^{-2\lambda_2 k^2 t}\, \epsilon^{2(N+1)}$$

yields, for $0 \le n \le N$,

$$|\xi_{\ell,k}|(t) \lesssim \epsilon^{N+1} k^{N+2} e^{-\lambda_2 k^2 t}$$



Moment errors

Error equations:

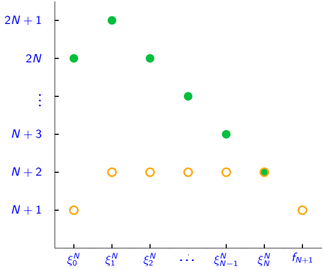
$$\begin{cases} \epsilon\partial_t\xi_{0,k} + a_0 ik\xi_{1,k} = 0, & \ell = 0; \\ \epsilon\partial_t\xi_{\ell,k} + a_l ik\xi_{\ell+1,k} + a_{\ell-1} ik\xi_{\ell-1,k} + \frac{1}{\epsilon}\xi_{\ell,k} = 0, & 1 \le \ell \le N-1; \\ \epsilon\partial_t\xi_{N,k} + a_{N-1} ik\xi_{N-1,k} + \frac{1}{\epsilon}\xi_{N,k} = -a_N ikf_{N+1,k}, & \ell = N. \end{cases}$$

For low frequency terms ($0 < |k|\epsilon \le \frac{1}{2}$): "Zoom-in procedure"

For $1 \le n \le N$,

$$\partial_t|\xi_{n,k}| + \frac{1}{\epsilon^2}|\xi_{n,k}|$$
$$\le \frac{k}{\epsilon}\left(a_{n-1}|\xi_{n-1,k}| + (1-\delta_{n,N})a_n|\xi_{n+1,k}| + \delta_{n,N}a_N|f_{N+1,k}|\right)$$
$$\lesssim \epsilon^N k^{N+3} e^{-\lambda_2 k^2 t}$$

$$\xrightarrow{\text{integrate over time}} |\xi_{n,k}|(t) \lesssim \epsilon^{N+2} k^{N+3} e^{-\lambda_2 k^2 t}$$



Moment errors

Error equations:

$$\begin{cases} \epsilon\partial_t\xi_{0,k} + a_0 ik\xi_{1,k} = 0, & \ell = 0; \\ \epsilon\partial_t\xi_{\ell,k} + a_l ik\xi_{\ell+1,k} + a_{\ell-1}ik\xi_{\ell-1,k} + \frac{1}{\epsilon}\xi_{\ell,k} = 0, & 1 \le \ell \le N-1; \\ \epsilon\partial_t\xi_{N,k} + a_{N-1}ik\xi_{N-1,k} + \frac{1}{\epsilon}\xi_{N,k} = -a_N ikf_{N+1,k}, & \ell = N. \end{cases}$$

For low frequency terms ($0 < |k|\epsilon \le \frac{1}{2}$): "Zoom-in procedure"

Consider

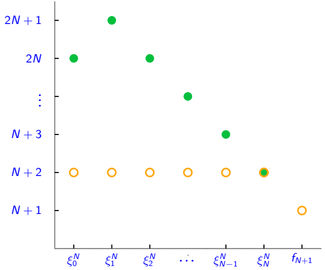
- smaller system $\{\xi_{n,k}\}_{n=0}^{N-1}$
- source term $\xi_{N,k}$

Using MME gives, for $0 \le n \le N-1$,

$$|\xi_{n,k}|(t) \lesssim \epsilon^{N+2} k^{N+4} e^{-\lambda_2 k^2 t}$$

In particular,

$$|\xi_{0,k}|(t) \lesssim \epsilon^{N+2} k^{N+4} e^{-\lambda_2 k^2 t}$$



Moment errors

Error equations:

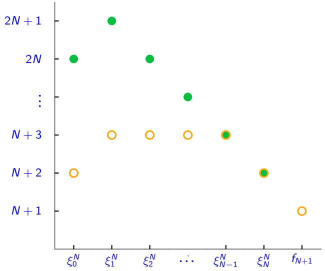
$$\begin{cases} \epsilon\partial_t\xi_{0,k} + a_0 ik\xi_{1,k} = 0, & \ell = 0; \\ \epsilon\partial_t\xi_{\ell,k} + a_l ik\xi_{\ell+1,k} + a_{\ell-1} ik\xi_{\ell-1,k} + \frac{1}{\epsilon}\xi_{\ell,k} = 0, & 1 \le \ell \le N-1; \\ \epsilon\partial_t\xi_{N,k} + a_{N-1} ik\xi_{N-1,k} + \frac{1}{\epsilon}\xi_{N,k} = -a_N ikf_{N+1,k}, & \ell = N. \end{cases}$$

For low frequency terms ($0 < |k|\epsilon \le \frac{1}{2}$): "Zoom-in procedure"

For $1 \le n \le N-1$,

$$\partial_t|\xi_{n,k}| + \frac{1}{\epsilon^2}|\xi_{n,k}|$$
$$\le \frac{k}{\epsilon}\left(a_{n-1}|\xi_{n-1,k}| + (1-\delta_{n,N})a_n|\xi_{n+1,k}| + \delta_{n,N}a_N|f_{N+1,k}|\right)$$
$$\lesssim \epsilon^{N+1}k^{N+5}e^{-\lambda_2 k^2 t}$$

$$\xrightarrow{\text{integrate over time}} |\xi_{n,k}|(t) \lesssim \epsilon^{N+3}k^{N+5}e^{-\lambda_2 k^2 t}$$



Moment errors

Error equations:

$$\begin{cases} \epsilon\partial_t\xi_{0,k} + a_0 ik\xi_{1,k} = 0, & \ell = 0; \\ \epsilon\partial_t\xi_{\ell,k} + a_l ik\xi_{\ell+1,k} + a_{\ell-1}ik\xi_{\ell-1,k} + \frac{1}{\epsilon}\xi_{\ell,k} = 0, & 1 \le \ell \le N-1; \\ \epsilon\partial_t\xi_{N,k} + a_{N-1}ik\xi_{N-1,k} + \frac{1}{\epsilon}\xi_{N,k} = -a_N ikf_{N+1,k}, & \ell = N. \end{cases}$$

For low frequency terms ($0 < |k|\epsilon \le \frac{1}{2}$): "Zoom-in procedure"

Consider

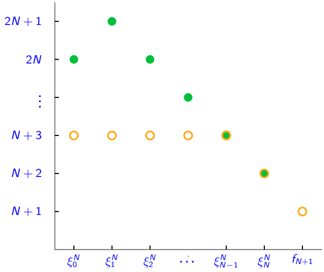
- smaller system $\{\xi_{n,k}\}_{n=0}^{N-2}$
- source term $\xi_{N-1,k}$

Using MME gives, for $0 \le n \le N-2$,

$$|\xi_{n,k}|(t) \lesssim \epsilon^{N+3}k^{N+6}e^{-\lambda_2 k^2 t}$$

In particular,

$$|\xi_{0,k}|(t) \lesssim \epsilon^{N+3}k^{N+6}e^{-\lambda_2 k^2 t}$$



Moment errors

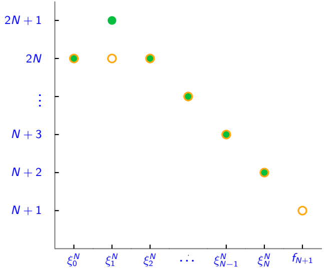
Error equations:

$$\begin{cases} \epsilon\partial_t\xi_{0,k} + a_0 ik\xi_{1,k} = 0, & \ell = 0; \\ \epsilon\partial_t\xi_{\ell,k} + a_l ik\xi_{\ell+1,k} + a_{\ell-1} ik\xi_{\ell-1,k} + \frac{1}{\epsilon}\xi_{\ell,k} = 0, & 1 \le \ell \le N-1; \\ \epsilon\partial_t\xi_{N,k} + a_{N-1} ik\xi_{N-1,k} + \frac{1}{\epsilon}\xi_{N,k} = -a_N ikf_{N+1,k}, & \ell = N. \end{cases}$$

For low frequency terms ($0 < |k|\epsilon \le \frac{1}{2}$): "Zoom-in procedure"

For $0 \le n \le 2$,

$$|\xi_{n,k}| \lesssim \epsilon^{2N} k^{3N} e^{-\lambda_2 k^2 t}$$



Moment errors

Error equations:

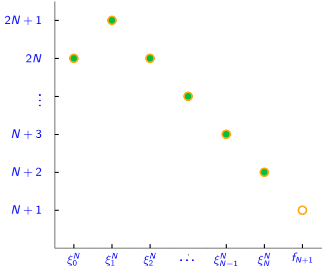
$$\begin{cases} \epsilon\partial_t\xi_{0,k} + a_0 ik\xi_{1,k} = 0, & \ell = 0; \\ \epsilon\partial_t\xi_{\ell,k} + a_l ik\xi_{\ell+1,k} + a_{\ell-1} ik\xi_{\ell-1,k} + \frac{1}{\epsilon}\xi_{\ell,k} = 0, & 1 \le \ell \le N-1; \\ \epsilon\partial_t\xi_{N,k} + a_{N-1} ik\xi_{N-1,k} + \frac{1}{\epsilon}\xi_{N,k} = -a_N ikf_{N+1,k}, & \ell = N. \end{cases}$$

For low frequency terms ($0 < |k|\epsilon \le \frac{1}{2}$): "Zoom-in procedure"

For $n = 1$,

$$\partial_t|\xi_{n,k}| + \frac{1}{\epsilon^2}|\xi_{n,k}|$$
$$\le \frac{k}{\epsilon}\left(a_{n-1}|\xi_{n-1,k}| + (1-\delta_{n,N})a_n|\xi_{n+1,k}| + \delta_{n,N}a_N|f_{N+1,k}|\right)$$
$$\lesssim \epsilon^{2N-1}k^{3N+1}e^{-\lambda_2 k^2 t}$$

$$\xrightarrow{\text{integrate over time}} |\xi_{1,k}|(t) \lesssim \epsilon^{2N+1}k^{3N+1}e^{-\lambda_2 k^2 t}$$



Moment errors

- Error equations:

$$\begin{cases} \epsilon\partial_t \xi_{0,k} + a_0 ik\xi_{1,k} = 0, & \ell = 0; \\ \epsilon\partial_t \xi_{\ell,k} + a_l ik\xi_{\ell+1,k} + a_{\ell-1} ik\xi_{\ell-1,k} + \frac{1}{\epsilon}\xi_{\ell,k} = 0, & 1 \le \ell \le N-1; \\ \epsilon\partial_t \xi_{N,k} + a_{N-1} ik\xi_{N-1,k} + \frac{1}{\epsilon}\xi_{N,k} = -a_N ikf_{N+1,k}, & \ell = N. \end{cases}$$

1. $k = 0$ terms: $\xi_{\ell,0} = 0$ for $0 \le \ell \le N$
2. High frequency terms: decay exponentially $\mathcal{O}(e^{-\frac{\lambda_1 t}{\epsilon^2}})$
3. Low frequency terms: method of induction + MME
   $\xi_{0,k} \sim \mathcal{O}(\epsilon^{2N})$, $\xi_{\ell,k} \sim \mathcal{O}(\epsilon^{2N+2-\ell})$

Theorem (Superconvergence in moments)

Given $g \in H^1(dx)$, there exists an absolute constant λ_1 s.t.

$$||\xi_\ell||_{L^2(dx)}(t) \le \begin{cases} C(\partial_x g)\sqrt{t}e^{-\frac{\lambda_1 t}{\epsilon^2}} + E(g, N, 2, t)\epsilon^{2N}, & \ell = 0 \\ C(\partial_x g)\sqrt{t}e^{-\frac{\lambda_1 t}{\epsilon^2}} + E(g, N, \ell, t)\epsilon^{2N+2-\ell}, & 1 \le \ell \le N. \end{cases}$$

Benefits from P_N to P_{N+1}

In main theorems,

$$||e^N||_{L^2(dxd\mu)}(t) \le B(g)e^{-\frac{\lambda_1 t}{\epsilon^2}} + C(\partial_x g)\sqrt{t}e^{-\frac{\lambda_1 t}{\epsilon^2}} + D(g,N,t)\epsilon^{N+1}.$$

$$||\xi_\ell^N||_{L^2(dx)}(t) \le \begin{cases} C(\partial_x g)\sqrt{t}e^{-\frac{\lambda_1 t}{\epsilon^2}} + E(g,N,2,t)\epsilon^{2N}, & \ell = 0 \\ C(\partial_x g)\sqrt{t}e^{-\frac{\lambda_1 t}{\epsilon^2}} + E(g,N,\ell,t)\epsilon^{2N+2-\ell}, & 1 \le \ell \le N. \end{cases}$$

Observation

1. $e^N = \mathcal{O}(\epsilon^{N+1})$
2. **a)** $\xi_0^N = \mathcal{O}(\epsilon^{2N})$
   b) other $\xi_\ell^N = \mathcal{O}(\epsilon^{2N+2-\ell})$

Theory

1. $e^N \le D(g,N,t)\epsilon^{N+1}$
2. **a)** $\xi_0^N \le E(g,N,2,t)\epsilon^{2N}$
   b) $\xi_\ell^N \le E(g,N,\ell,t)\epsilon^{(2N+2-\ell)}$

Using the error estimates to indicate the benefit from P_N to P_{N+1}:

$$\frac{||e^{N+1}||}{||e^N||} \sim \frac{D(g, N+1, t)\epsilon^{N+2}}{D(g, N, t)\epsilon^{N+1}} = \frac{D(g, N+1, t)}{D(g, N, t)}\epsilon = \mathcal{O}(\epsilon),$$

if $D(g, N, t)$ does NOT grow fast w.r.t. N. Similarly,

$$\frac{||\xi_\ell^{N+1}||}{||\xi_\ell^N||} \sim \frac{E(g, N+1, \ell, t)}{E(g, N, \ell, t)}\epsilon^2 = \mathcal{O}(\epsilon^2).$$

Observation

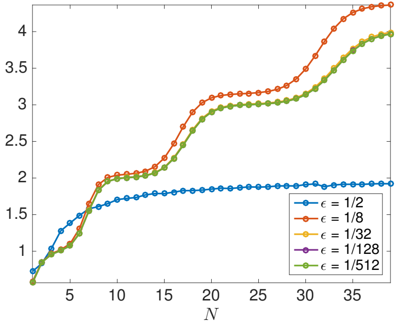
1. $e^N = \mathcal{O}(\epsilon^{N+1})$
2. a) $\xi_0^N = \mathcal{O}(\epsilon^{2N})$
   b) other $\xi_\ell^N = \mathcal{O}(\epsilon^{2N+2-\ell})$

Theory

1. $e^N \leq D(g, N, t)\epsilon^{N+1}$
2. a) $\xi_0^N \leq E(g, N, 2, t)\epsilon^{2N}$
   b) $\xi_\ell^N \leq E(g, N, \ell, t)\epsilon^{(2N+2-\ell)}$

Benefits of increasing N: $||e^N||_{L^2(d\mu dx)}$

$$\frac{\|e^{N+1}\|}{\|e^N\|_\epsilon} < 5$$

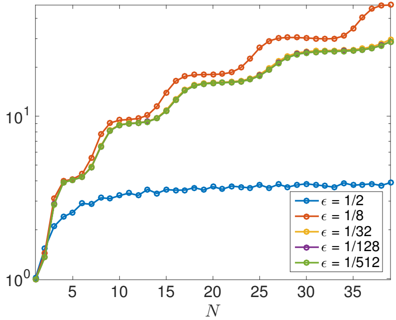


(a) $\frac{||e^{N+1}||}{||e^N||_\epsilon}$

Errors of P_N equations, $N = 1, \cdots, 40$.

Benefits of increasing N: $||\xi_0^N||_{L^2(dx)}$

$$\frac{||\xi_0^{N+1}||}{||\xi_0^N||\epsilon^2} < 50$$

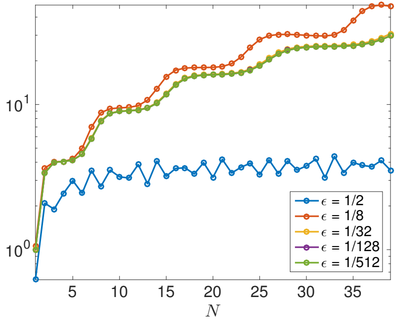


(a) $\frac{||\xi_0^{N+1}||}{||\xi_0^N||\epsilon^2}$

Errors of f_0^N of P_N equations, $N = 1, \cdots, 40$.

Benefits of increasing N: $||\xi_1^N||_{L^2(dx)}$

$$\frac{||\xi_1^{N+1}||}{||\xi_1^N||\epsilon^2} < 50$$

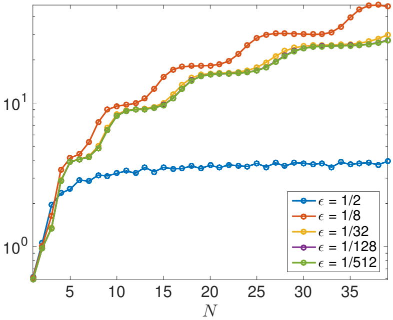


(a) $\frac{||\xi_1^{N+1}||}{||\xi_1^N||\epsilon^2}$

Errors of f_1^N of P_N equations, $N = 1, \cdots, 40$.

Benefits of increasing N: $||\xi_2^N||_{L^2(dx)}$

$$\frac{||\xi_2^{N+1}||}{||\xi_2^N||\epsilon^2} < 50$$



(a) $\frac{||\xi_2^{N+1}||}{||\xi_2^N||\epsilon^2}$

Errors of f_2^N of P_N equations, $N = 1, \cdots, 40$.

The sum of the series

All these constants in the estimates are multiples of the following sum

$$a_n(s) := \sum_{k>0} \left\{ k^{2n} e^{-k^2 s} \right\},$$

with some n, $s = 2\lambda_2 t$ and the constant λ_2, which has the same **step behavior** as in the previous numerical benefits, and has the same growth rate as function

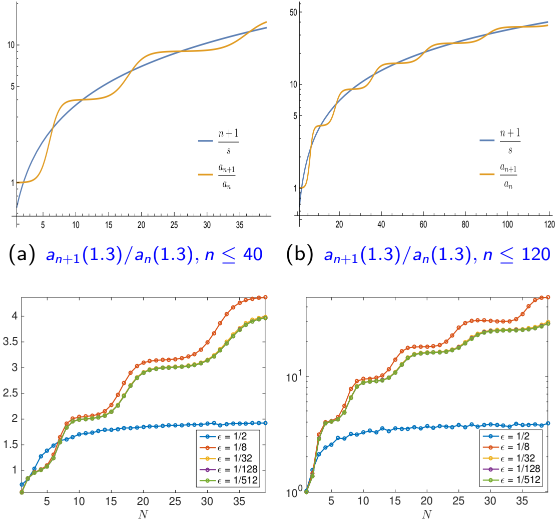
$$a_n(s) \sim \frac{n!}{s^n},$$

and

$$a_{n+1}(s)/a_n(s) \sim \frac{(n+1)!}{s^{n+1}} \Big/ \frac{n!}{s^n} = \frac{n+1}{s}.$$

We also have some analytic results to support this numerical observation.

The sum of the series



(a) $a_{n+1}(1.3)/a_n(1.3)$, $n \leq 40$ (b) $a_{n+1}(1.3)/a_n(1.3)$, $n \leq 120$

(c) $\frac{||e^{N+1}||}{||e^N||\epsilon}(1)$ (d) $\frac{||\xi_0^{N+1}||}{||\xi_0^N||\epsilon^2}(1)$

Results: anisotropic i.c.

Numerical results

Super convergence properties relate to the highest non-zero moment of initial condition g. Consider

$$g(x,\mu) = \sum_{\ell=0}^{m_0} g_\ell(x) p_\ell(\mu).$$

m_0	numerical	theoretical
$m_0 = 0, 1$	$\min\{2N+\ell, 2N+2-\ell\}$	$\min\{2N+\ell, 2N+2-\ell\}$
$2 \leq m_0 < N+1$	$\min\{2N+2-m_0+\ell, 2N+2-\ell\}$	$\min\{2N+1-m_0+\ell, 2N+2-\ell\}$
$m_0 \geq N+1$	$\min\{N+1+\ell, 2N+2-\ell\}$	$\min\{N+\ell, 2N+2-\ell\}$

Table: Numerical and theoretical orders of convergence rates for the ℓ^{th} moment.

Numerical results

m_0	numerical	theoretical
$m_0 = 0, 1$	$\min\{2N+\ell, 2N+2-\ell\}$	$\min\{2N+\ell, 2N+2-\ell\}$
$2 \leq m_0 < N+1$	$\min\{2N+2-m_0+\ell, 2N+2-\ell\}$	$\min\{2N+1-m_0+\ell, 2N+2-\ell\}$
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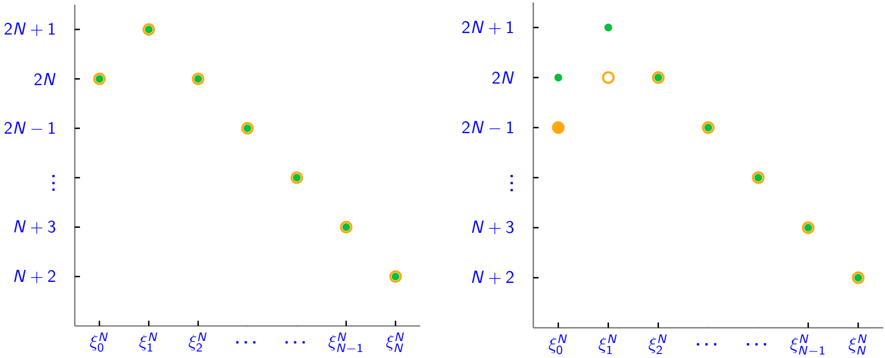

Left: $m_0 = 0$ or 1; right: $m_0 = 2$.

Numerical results

m_0	numerical	theoretical
$m_0 = 0, 1$	$\min\{2N+\ell, 2N+2-\ell\}$	$\min\{2N+\ell, 2N+2-\ell\}$
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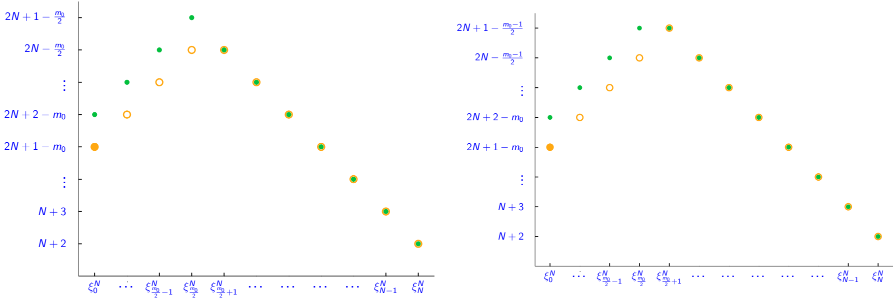

$2 \leq m_0 < N+1$

Left: m_0 even; right: m_0 odd.

Numerical results

m_0	numerical	theoretical
$m_0 = 0, 1$	$\min\{2N+\ell, 2N+2-\ell\}$	$\min\{2N+\ell, 2N+2-\ell\}$
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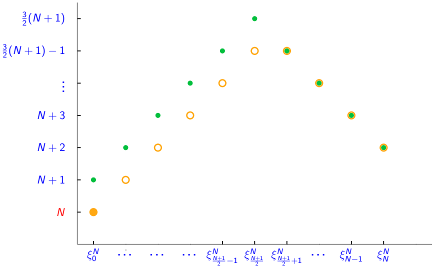

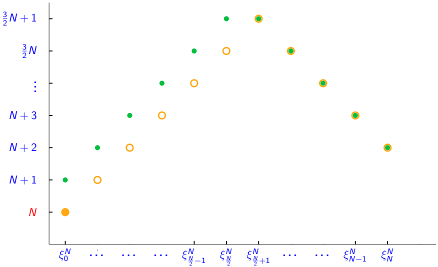

$m_0 \geq N+1$
Left: N odd; right: N even.

Summary

- A multiscale convergence property for the error of P_N method, which is $\mathcal{O}(\epsilon^{N+1})$, for linear kinetic equations with isotropic scattering, no absorption, no source, and isotropic initial condition.

- Super-convergence rate in the spectral approximation for each moment with P_N method.

- Asymptotic estimates for moments, i.e., f_ℓ and f_ℓ^N are $\mathcal{O}(\epsilon^\ell)$.

- Future work:
  - optimal convergence rate for cases with anisotropic initial condtions
  - realistic domain with boundary conditions
  - non-zero absorption and sources
  - spatially dependent scattering and anisotropic scattering
  - alternative angular discretizations and nonlinear systems
  - error estimate for multiscale hybrid methods

Backup slides

Initial estimates on f_ℓ

- **Method of modified energy (MME)**: $(u \Leftarrow f)$

$$\partial_t \left(H_0^k(f) + h_\gamma^k(f) \right) \leq -\frac{2}{\epsilon^2} H_3^k(f) - \sum_{\ell=0}^{2} c_{\gamma,\ell} |f_{\ell,k}|^2$$

where

$$c_{\gamma,0} = \frac{\gamma k}{16\epsilon}, \qquad c_{\gamma,1} = \frac{1}{\epsilon^2} - \frac{\gamma k}{4\epsilon} - \frac{3\gamma}{8\epsilon^3 k}, \qquad c_{\gamma,2} = \frac{1}{\epsilon^2} - \frac{\gamma k}{5\epsilon}.$$

For each k, choose suitable γ, and find positive lower bounds on $\{c_{\gamma,i}\}_{i=0}^2$.

Initial estimates on f_ℓ

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$$\partial_t \left(H_0^k(f) + h_\gamma^k(f) \right) \le -\frac{2}{\epsilon^2} H_3^k(f) - \sum_{\ell=0}^{2} c_{\gamma,\ell} |f_{\ell,k}|^2$$

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For each k, choose suitable γ, and find positive lower bounds on $\{c_{\gamma,i}\}_{i=0}^2$.

(i) High frequency: $(k\epsilon > 1/2)$

$$\gamma := \gamma^{\mathrm{high}} := \frac{16}{29} \frac{1}{k\epsilon} < \frac{32}{29}.$$

Then

$$\partial_t \left(H_0^k(f) + h_\gamma^k(f) \right) \le -\frac{2\lambda_1}{\epsilon^2} \left(H_0^k(f) + h_\gamma^k(f) \right)$$

$$\xrightarrow{\text{integrate in time}} \left(H_0^k(f) + h_\gamma^k(f) \right)(t) \le e^{-\frac{2\lambda_1 t}{\epsilon^2}} \left(H_0^k(g) + h_\gamma^k(g) \right)$$

$$\xrightarrow{\text{equivalence}} H_0^k(f)(t) < 6 H_0^k(g)\, e^{-\frac{2\lambda_1 t}{\epsilon^2}}.$$

Initial estimates on f_ℓ

- **Method of modified energy (MME)**: $(u \Leftarrow f)$

$$\partial_t \left(H_0^k(f) + h_\gamma^k(f)\right) \le -\frac{2}{\epsilon^2} H_3^k(f) - \sum_{\ell=0}^{2} c_{\gamma,\ell} |f_{\ell,k}|^2$$

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For each k, choose suitable γ, and find positive lower bounds on $\{c_{\gamma,i}\}_{i=0}^2$.

(ii) Low frequency: $(0 < k\epsilon \le \frac{1}{2})$

$$\gamma := \gamma^{\mathrm{low}} := \frac{64}{29} k\epsilon \le \frac{32}{29}.$$

Then

$$\partial_t \left(H_0^k(f) + h_\gamma^k(f)\right) \le -\frac{2\lambda_2}{45} k^2 \left(H_0^k(f) + h_\gamma^k(f)\right)$$

$$\xrightarrow{\text{integrate in time}} \left(H_0^k(f) + h_\gamma^k(f)\right)(t) \le e^{-2\lambda_2 k^2 t} \left(H_0^k(g) + h_\gamma^k(g)\right)$$

$$\xrightarrow{\text{equivalence}} H_0^k(f)(t) < 6 H_0^k(g)\, e^{-2\lambda_2 k^2 t}.$$

Initial estimates on f_ℓ

- **Method of modified energy (MME)**: $(u \Leftarrow f)$

  (i) High frequency: $(k\epsilon > 1/2)$

$$H_0^k(f)(t) < 6H_0^k(g)\, e^{-\frac{2\lambda_1 t}{\epsilon^2}}$$

  (ii) Low frequency: $(0 < k\epsilon \leq \frac{1}{2})$

$$H_0^k(f)(t) < 6H_0^k(g)\, e^{-2\lambda_2 k^2 t}$$

$$H_0^k(f) := \frac{1}{2}\sum_{\ell=0}^{\infty} |f_{\ell,k}|^2$$

$\Downarrow$

$f_{\ell,k}$

$\Downarrow$ sum over k

f_ℓ

Initial estimates on f_ℓ

- **Method of modified energy (MME)**: $(u \Leftarrow f)$

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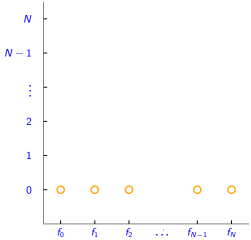


Initial estimate:

$$\|f_\ell\|^2_{L^2(dx)}(t) \leq \overbrace{12 \sum_{|k|\epsilon > 1/2} H_0^k(g) e^{-\frac{2\lambda_1 t}{\epsilon^2}}}^{\text{high frequencies}} + \underbrace{C(g,0,t)^2}_{\text{low frequencies}}$$

with

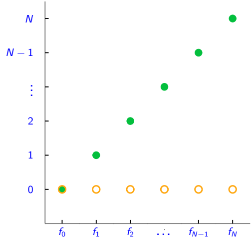
$$C(g,0,t) = \left[24 \max_{k>0} H_0^k(g) \sum_{k>0} e^{-2\lambda_2 k^2 t}\right]^{\frac{1}{2}}.$$

Initial estimates on f_ℓ

- **Method of modified energy (MME)**: $(u \Leftarrow f)$

Initial estimate:

$$\|f_\ell\|^2_{L^2(dx)}(t) \le 12 \sum_{|k|\epsilon > 1/2} H_0^k(g) e^{-\frac{2\lambda_1 t}{\epsilon^2}} + C(g,0,t)^2$$



- **Next goal:** f_ℓ is bounded by **a finer estimate**:

$$\|f_\ell\|^2_{L^2(dx)}(t) \le \overbrace{12 \sum_{|k|\epsilon > 1/2} H_0^k(g) e^{-\frac{2\lambda_1 t}{\epsilon^2}}}^{\text{high frequencies}} + \underbrace{C(g,\ell,t)^2 \epsilon^{2\ell}}_{\text{low frequencies}}$$

with

$$C(g,\ell,t) = \left[24 \max_{k>0} H_0^k(g) \sum_{k>0} (Ak)^{2\ell} e^{-2\lambda_2 k^2 t} \right]^{\frac{1}{2}}.$$

Stability error

- Stability error equation:

$$\begin{cases} \epsilon\partial_t\xi_{0,k} + a_0 ik\xi_{1,k} = 0, & \ell = 0; \\ \epsilon\partial_t\xi_{\ell,k} + a_\ell ik\xi_{\ell+1,k} + a_{\ell-1} ik\xi_{\ell-1,k} + \frac{1}{\epsilon}\xi_{\ell,k} = 0, & 1 \le \ell \le N-1; \\ \epsilon\partial_t\xi_{N,k} + a_{N-1} ik\xi_{N-1,k} + \frac{1}{\epsilon}\xi_{N,k} = -a_N ikf_{N+1,k}, & \ell = N. \end{cases}$$

Stability error

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$$\partial_t \sum_{\ell=0}^{N} |\xi_{\ell,k}|^2 + \frac{2}{\epsilon^2}\sum_{\ell=1}^{N} |\xi_{\ell,k}|^2 \le \frac{a_N|k|}{\epsilon}|\xi_{N,k}||f_{N+1,k}|$$

$$\le \frac{1}{2\epsilon^2}|\xi_{N,k}|^2 + \frac{k^2}{6}|f_{N+1,k}|^2$$

Stability error

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$$\partial_t \sum_{\ell=0}^{N} |\xi_{\ell,k}|^2 + \frac{2}{\epsilon^2}\sum_{\ell=1}^{N}|\xi_{\ell,k}|^2 \le \frac{a_N|k|}{\epsilon}|\xi_{N,k}||f_{N+1,k}| \le \frac{1}{2\epsilon^2}|\xi_{N,k}|^2 + \frac{k^2}{6}|f_{N+1,k}|^2$$

$$\partial_t\left(H_0^k(\xi) + h_\gamma^k(\xi)\right) \le -\frac{2}{\epsilon^2}H_3^k(\xi) - \sum_{\ell=0}^{2} c_{\gamma,\ell}|\xi_{\ell,k}|^2 + \frac{1}{\epsilon^2}\frac{1}{2}|\xi_{N,k}|^2 + \frac{k^2}{6}|f_{N+1,k}|^2$$

$$\le -\frac{1}{\epsilon^2}H_3^k(\xi) - \sum_{\ell=0}^{2} c_{\gamma,\ell}|\xi_{\ell,k}|^2 + \frac{k^2}{6}|f_{N+1,k}|^2,$$

1. for f:

$$\partial_t\left(H_0^k(f) + h_\gamma^k(f)\right) \le -\frac{2}{\epsilon^2}H_3^k(f) - \sum_{\ell=0}^{2} c_{\gamma,\ell}|f_{\ell,k}|^2$$

2. $\xi_{\ell,k}(0) = 0$

Stability error

- Stability error equation:

$$\begin{cases} \epsilon\partial_t\xi_{0,k} + a_0 ik\xi_{1,k} = 0, & \ell = 0; \\ \epsilon\partial_t\xi_{\ell,k} + a_\ell ik\xi_{\ell+1,k} + a_{\ell-1} ik\xi_{\ell-1,k} + \frac{1}{\epsilon}\xi_{\ell,k} = 0, & 1 \le \ell \le N-1; \\ \epsilon\partial_t\xi_{N,k} + a_{N-1} ik\xi_{N-1,k} + \frac{1}{\epsilon}\xi_{N,k} = -a_N ikf_{N+1,k}, & \ell = N. \end{cases}$$

$$\partial_t \left(H_0^k(\xi) + h_\gamma^k(\xi) \right) \le -\frac{1}{\epsilon^2} H_3^k(\xi) - \sum_{\ell=0}^{2} c_{\gamma,\ell} |\xi_{\ell,k}|^2 + \frac{k^2}{6} |f_{N+1,k}|^2,$$

(i) High frequency: With estimate on source of the error coefficient system

$$\frac{k^2}{6} |f_{N+1,k}|^2(t) \le 2k^2 H_0^k(g)\, e^{-\frac{2\lambda_1 t}{\epsilon^2}}$$

$$\xrightarrow{\text{MME}} \quad H_0^k(\xi)(t) \le 6t\, k^2 H_0^k(g)\, e^{-\frac{2\lambda_1 t}{\epsilon^2}} = 6t\, H_0^k(\partial_x g)\, e^{-\frac{2\lambda_1 t}{\epsilon^2}}.$$

Stability error

- Stability error equation:

$$\begin{cases} \epsilon\partial_t\xi_{0,k} + a_0 ik\xi_{1,k} = 0, & \ell = 0; \\ \epsilon\partial_t\xi_{\ell,k} + a_\ell ik\xi_{\ell+1,k} + a_{\ell-1}ik\xi_{\ell-1,k} + \frac{1}{\epsilon}\xi_{\ell,k} = 0, & 1 \le \ell \le N-1; \\ \epsilon\partial_t\xi_{N,k} + a_{N-1}ik\xi_{N-1,k} + \frac{1}{\epsilon}\xi_{N,k} = -a_N ikf_{N+1,k}, & \ell = N. \end{cases}$$

$$\partial_t\left(H_0^k(\xi) + h_\gamma^k(\xi)\right) \le -\frac{1}{\epsilon^2}H_3^k(\xi) - \sum_{\ell=0}^{2} c_{\gamma,\ell}|\xi_{\ell,k}|^2 + \frac{k^2}{6}|f_{N+1,k}|^2,$$

(ii) Low frequency: With estimate on source of the error coefficient system

$$|f_{N+1,k}|(t) \le C_{N+1}^k \epsilon^{N+1} k^{N+1} e^{-\lambda_2 k^2 t}$$

$$\stackrel{\text{MME}}{\Longrightarrow} \quad \left(H_0^k(\xi) + h_\gamma^k(\xi)\right)(t) \le \frac{k^2}{6}e^{-2\lambda_2 k^2 t}\int_0^t e^{2\lambda_2 k^2 s}|f_{N+1,k}|^2\, ds$$

$$\le \frac{t}{6}(C_{N+1}^k)^2 k^{2(N+2)} e^{-2\lambda_2 k^2 t}\, \epsilon^{2(N+1)}.$$

Stability error

- Stability error equation:

$$\begin{cases} \epsilon\partial_t\xi_{0,k} + a_0 ik\xi_{1,k} = 0, & \ell = 0; \\ \epsilon\partial_t\xi_{\ell,k} + a_\ell ik\xi_{\ell+1,k} + a_{\ell-1} ik\xi_{\ell-1,k} + \frac{1}{\epsilon}\xi_{\ell,k} = 0, & 1 \le \ell \le N-1; \\ \epsilon\partial_t\xi_{N,k} + a_{N-1} ik\xi_{N-1,k} + \frac{1}{\epsilon}\xi_{N,k} = -a_N ikf_{N+1,k}, & \ell = N. \end{cases}$$

$$\partial_t\left(H_0^k(\xi) + h_\gamma^k(\xi)\right) \le -\frac{1}{\epsilon^2}H_3^k(\xi) - \sum_{\ell=0}^{2} c_{\gamma,\ell}|\xi_{\ell,k}|^2 + \frac{k^2}{6}|f_{N+1,k}|^2,$$

(i) High frequency: $H_0^k(\xi)(t) \le 6t\, H_0^k(\partial_x g)\, e^{-\frac{2\lambda_1 t}{\epsilon^2}}$

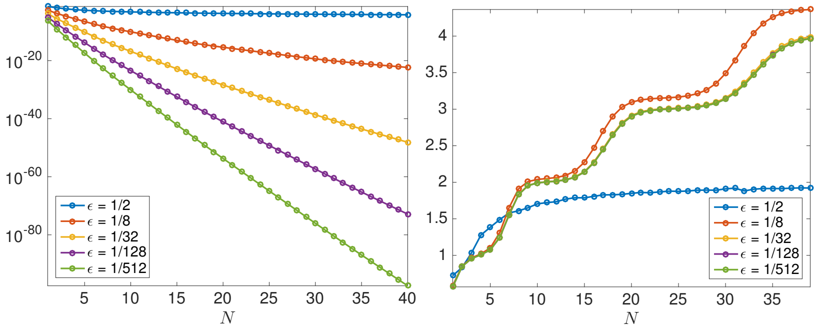
(ii) Low frequency: $H_0^k(\xi)(t) \le \frac{t}{2}[C_{N+1}^k]^2 k^{2(N+2)} e^{-2\lambda_2 k^2 t}\, \epsilon^{2(N+1)}$

Lemma

Given isotropic initial condition $g \in H^1(dx)$,

$$||\xi||_{L^2(d\mu dx)}(t) \le C(\partial_x g)\sqrt{t}e^{-\frac{\lambda_1 t}{\epsilon^2}} + \frac{\sqrt{t}}{A}C(g, N+2, t)\epsilon^{N+1}.$$

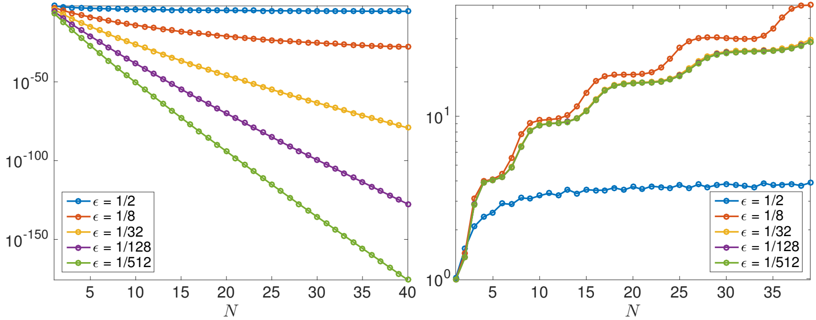
Benefits of increasing N: $||e^N||_{L^2(d\mu dx)}$



(a) $||e^N||$, in logarithm scale (b) $\frac{||e^{N+1}||}{||e^N||\epsilon}$

Errors of P_N equations, $N = 1, \cdots, 40$.

Benefits of increasing N: $||\xi_0^N||_{L^2(dx)}$

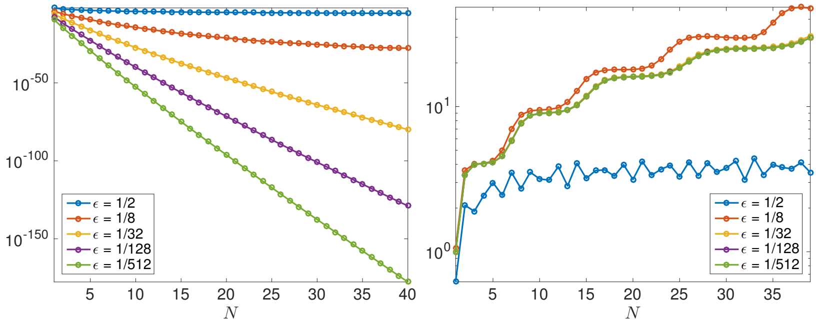


(a) $||\xi_0^N||$, in logarithm scale

(b) $\frac{||\xi_0^{N+1}||}{||\xi_0^N||\epsilon^2}$

Errors of f_0^N of P_N equations, $N = 1, \cdots, 40$.

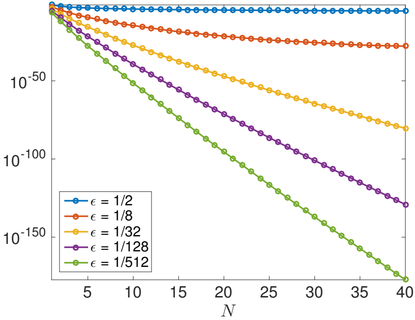
Benefits of increasing N: $||\xi_1^N||_{L^2(dx)}$



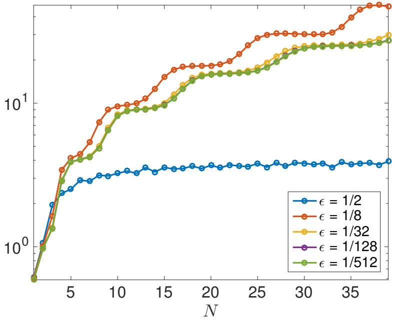
(a) $||\xi_1^N||$, in logarithm scale (b) $\frac{||\xi_1^{N+1}||}{||\xi_1^N||\epsilon^2}$

Errors of f_1^N of P_N equations, $N = 1, \cdots, 40$.

Benefits of increasing N: $||\xi_2^N||_{L^2(dx)}$



(a) $||\xi_2^N||$, in logarithm scale



(b) $\frac{||\xi_2^{N+1}||}{||\xi_2^N||\epsilon^2}$

Errors of f_2^N of P_N equations, $N = 1, \cdots, 40$.