Third order Maximum-Principle-Satisfying Direct DG methods for convection diffusion equations on unstructured triangular mesh

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Object: design high order Maximum-Principle-Satisfying DG method

$$u_t + \nabla \cdot F(u) - \nabla \cdot (A(u)\nabla u) = 0$$

- PDE (continuous): The solution satisfies the maximum principle:

$$m \le u(x, y, t) \le M, \qquad (x, y) \in \Omega, t > 0,$$

  where, m and M are the minimum and maximum of the initial and boundary data.

- Schemes (discrete): We expect the numerical solution to satisfy the discrete maximum principle at all time level t^n:

$$m \le u_K^n(x, y) \le M, \qquad (x, y) \in K \in \mathcal{T}_h \subseteq \Omega.$$

  - Strong stability result in the L^∞ sense
  - Consistency to physical meanings (e.g. preserve positivity of density profile)

M-P-S high order DG methods for hyperbolic PDEs

$$u_t + \nabla \cdot F(u) = 0 \quad \Longrightarrow \quad m \le u_K^n(x,y) \le M$$

For convection term:

M-P-S DGM on conservation laws

- X. Zhang and C.-W. Shu (2010a). "On maximum-principle-satisfying high order schemes for scalar conservation laws". In: *J. Comput. Phys.*
- X. Zhang and C.-W. Shu (2011). "Maximum-principle-satisfying and positivity-preserving high-order schemes for conservation laws: survey and new developments". In: *Proc. R. Soc. A*
- X. Zhang, Y. Xia, and C.-W. Shu (2012). "Maximum-principle-satisfying and positivity-preserving high order discontinuous Galerkin schemes for conservation laws on triangular meshes". In: *J. Sci. Comput.*

$$u_t + \nabla \cdot F(u) - \nabla \cdot (A(u)\nabla u) = 0 \quad \Longrightarrow \quad m \le u_K^n(x,y) \le M$$

For diffusion term:

- X. Zhang Y. Zhang and C.-W. Shu (2013). "Maximum-principle-satisfying second order discontinuous Galerkin schemes for convection-diffusion equations on triangular meshes". In: *J. Comput. Phys.*
- Z. Chen, H. Huang, and J. Yan (2016). "Third order maximum-principle-satisfying direct discontinuous Galerkin methods for time dependent convection diffusion equations on unstructured triangular meshes". In: *Journal of Computational Physics* 308, pp. 198–217

Outline

1. Reviews
   - Discontinuous Galerkin methods
   - General framework of M-P-S schemes
   - Introduction to Direct DG method and its variations
2. M-P-S DDG schemes for triangular mesh
   - Methods on rectangular mesh
   - New ideas designed for unstructured mesh
   - Extension to nonlinear diffusion equations
   - Algorithm
3. Numerical examples
4. Conclusions and future works

Joint work with: Jue Yan (Iowa State University)
Hongying Huang (Zhejiang Ocean University, China)

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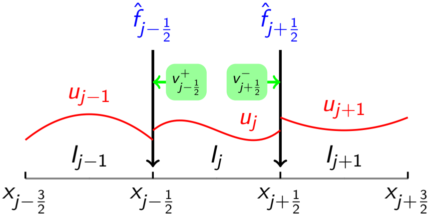
Review of discontinuous Galerkin (DG) methods

$$\boxed{u_t + f(u)_x = 0} \Longrightarrow \boxed{\int_{I_j} u_t v\, dx + \int_{I_j} f(u)_x v\, dx = 0}$$

$$\xrightarrow{\text{Integration by part}} \boxed{\int_{I_j} u_t v\, dx - \int_{I_j} f(u) v_x\, dx + [f(u)v]\Big|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} = 0}$$

DG solution:



$$\boxed{\int_{I_j} u_t v\, dx - \int_{I_j} f(u) v_x\, dx + \hat{f}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0}$$

Outline

Major steps to prove M-P-S schemes

Assumption:

$$|u(x,t^n) - u_j^n(x)| \le C\Delta x^3 \text{ and } m \le u_j^n(x) \le M,\ \forall x \in I_j$$

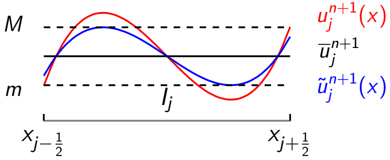
- **Step 1:** evolve in time once and estimate the solution average

$$\overline{u}_j^{n+1} = \frac{1}{\Delta x_j}\int_{I_j} u_j^{n+1}(x)\,dx$$

$$\boxed{m \le \overline{u}_j^{n+1} \le M} \quad \longleftarrow \quad \text{difficult!}$$

- **Step 2:** modify the solution by adding a limiter to make sure

$$\boxed{m \le \tilde{u}_j^{n+1}(x) \le M} \quad \longleftarrow \quad \text{easy}$$



$$\tilde{u}_j^{n+1}(x) = \theta(u_j^{n+1}(x) - \overline{u}_j^{n+1}) + \overline{u}_j^{n+1}$$

$$\theta = \min\left\{1, \left|\frac{M-\overline{u}_j^{n+1}}{M_j-\overline{u}_j^{n+1}}\right|, \left|\frac{m-\overline{u}_j^{n+1}}{m_j-\overline{u}_j^{n+1}}\right|\right\}$$

Difficulty to obtain high order M-P-S for diffusion equation

Hyperbolic	**Parabolic**
$u_t + f(u)_x = 0$	$u_t = u_{xx}$
⬇	⬇
$\int_{I_j} u_t v\, dx - \int_{I_j} f(u) v_x\, dx = -\hat{f}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + \hat{f}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}}$	$\int_{I_j} u_t v\, dx + \int_{I_j} u_x v_x\, dx = (\widehat{u_x})_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - (\widehat{u_x})_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}}$
⬇	⬇
$\overline{u}_j^{n+1} = \overline{u}_j^n + \frac{\Delta t}{\Delta x}\left(\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}\right)$	$\overline{u}_j^{n+1} = \overline{u}_j^n + \frac{\Delta t}{\Delta x}\left((\widehat{u_x})_{j+\frac{1}{2}} - (\widehat{u_x})_{j-\frac{1}{2}}\right)$
solution values on cell boundaries	solution derivative values on cell boundaries

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Outline

Direct DG (DDG) method as a diffusion solver

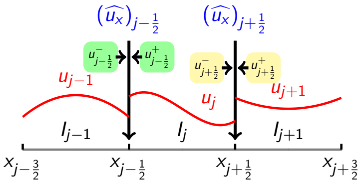
Parabolic

$$\boxed{u_t = u_{xx}} \implies \boxed{\int_{I_j} u_t v\, dx + \int_{I_j} u_x v_x\, dx - (\widehat{u_x})_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + (\widehat{u_x})_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0}$$

At cell boundaries:

$$[u] = u^+ - u^-$$

$$\overline{u} = \frac{u^+ + u^-}{2}$$



$$\boxed{\widehat{u_x}\Big|_{x_{j\pm\frac{1}{2}}} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x} + \beta_1 \Delta x [u_{xx}] + \beta_2 \Delta x^3 [u_{xxxx}] + \cdots}\,^*$$

*H. Liu and J. Yan (2009). "The Direct Discontinuous Galerkin (DDG) Methods for Diffusion Problems". In: *SIAM J. Numer. Anal.*

DDG method with interface correction

$$\int_{I_j} u_t v\,dx + \int_{I_j} u_x v_x\,dx - (\widehat{u_x} v)\Big|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} + \boxed{\sigma\left([u](\widetilde{v_x})_{j+\frac{1}{2}} + [u](\widetilde{v_x})_{j-\frac{1}{2}}\right)} = 0$$

- DDG: $\sigma = 0$, $\quad \widehat{u_x} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x} + \beta_1 \Delta x [u_{xx}] + \beta_3 \Delta x^3 [u_{xxxx}] + \cdots$
- DDG with interface correction: $\sigma = 1$,

$$\begin{cases} \widehat{u_x} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x} + \beta_1 \Delta x [u_{xx}] \\ \widetilde{v_x} = \overline{v_x} \end{cases}$$

DDG method with interface correction

$$\int_{I_j} u_t v\,dx + \int_{I_j} u_x \cancel{v_x}\,dx - (\widehat{u_x} v)\Big|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} + \boxed{\sigma\left([u]\cancel{(\widetilde{v_x})}_{j+\frac{1}{2}} + [u]\cancel{(\widetilde{v_x})}_{j-\frac{1}{2}}\right)} = 0$$

- DDG: $\sigma = 0$, $\quad \widehat{u_x} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x} + \beta_1 \Delta x [u_{xx}] + \beta_3 \Delta x^3 [u_{xxxx}] + \cdots$
- DDG with interface correction: $\sigma = 1$,

$$\begin{cases} \widehat{u_x} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x} + \beta_1 \Delta x [u_{xx}] \\ \widetilde{v_x} = \overline{v_x} \end{cases}$$

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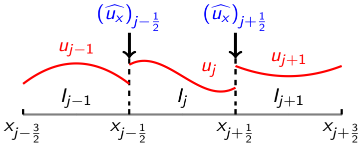
Solution average evolution

Take $v = 1$

$$\frac{d}{dt}\int_{I_j} u\,dx = (\widehat{u_x})\Big|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \implies \frac{d}{dt}\overline{u} = \frac{1}{\Delta x}\left\{(\widehat{u_x})_{j+\frac{1}{2}} - (\widehat{u_x})_{j-\frac{1}{2}}\right\}$$

Euler forward in time evolution

$$\overline{u}_j^{n+1} = \overline{u}_j^n + \frac{\Delta t}{\Delta x}\left\{(\widehat{u_x})_{j+\frac{1}{2}} - (\widehat{u_x})_{j-\frac{1}{2}}\right\}$$
$$= H(u_{j-1}^n(x), u_j^n(x), u_{j+1}^n(x))$$



Goal

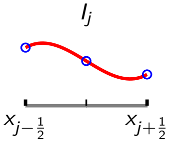
$$m \le u_{j-1}^n(x), u_j^n(x), u_{j+1}^n(x) \le M \implies m \le \overline{u}_j^{n+1} \le M$$

Outline

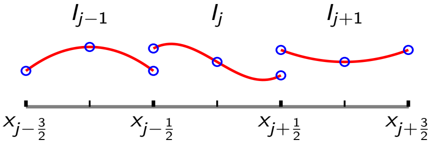
Previous method to bound solution average

Previous idea

Pick $u^+_{j-\frac{1}{2}}, u_j, u^-_{j+\frac{1}{2}}$ as degrees of freedom and rewrite the quadratic polynomial $u^u_j(x)$ in the Lagrange format.

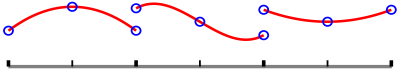


$$\begin{aligned}\overline{u}_j^{n+1} &= H(u^n_{j-1}(x), u^n_j(x), u^{n+1}_j(x)) \\ &= H(u^+_{j-\frac{3}{2}}, u_{j-1}, u^-_{j-\frac{1}{2}}, u^+_{j-\frac{1}{2}}, u_j, u^-_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}}, u_{j+1}, u^-_{j+\frac{3}{2}})\end{aligned}$$



M-P-S 3rd order DDG schemes

$$\overline{u}_j^{n+1} = H(u^+_{j-\frac{3}{2}}, u_{j-1}, u^-_{j-\frac{1}{2}}, u^+_{j-\frac{1}{2}}, u_j, u^-_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}}, u_{j+1}, u^-_{j+\frac{3}{2}})$$
$$= H(\uparrow,\uparrow,\uparrow,\uparrow,\uparrow,\uparrow,\uparrow,\uparrow,\uparrow) \text{ convex combination}$$



Given $m \leq u_j^n(x) \leq M$, we have

$$m = H(m, \cdots, m) \leq \overline{u}_j^{n+1} \leq H(M, \cdots, M) = M$$

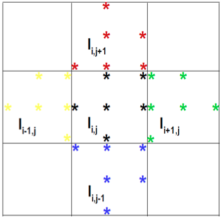
Conditions needed:

$$\beta_0 \geq \frac{3}{2} \qquad \frac{1}{8} \leq \beta_1 \leq \frac{1}{4}$$

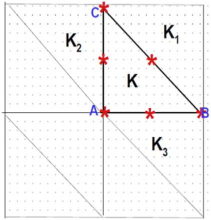
$$\widehat{u_x} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x} + \beta_1 \Delta x [u_{xx}]$$

Same results holds for SSP Runge Kutta methods.

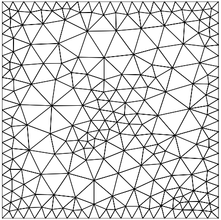
Difficulty for unstructured triangular mesh



(a) rectangular mesh



(b) triangular mesh



(c) unstructured triangular mesh

For any shape triangle, it is very hard to identify the 6 degrees of freedom to rewrite the quadratic polynomial $u_K^n(x, y)$ to bound

$$m \leq \overline{u}_K^{n+1} \leq M$$

Outline

New idea for triangular mesh

$$u_t - \Delta u = 0$$

DDG schemes:

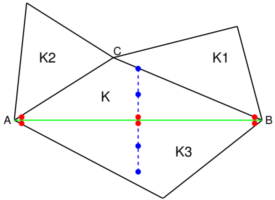
$$\int_K u_t v \; dxdy + \int_K \nabla u \cdot \nabla v \; dxdy - \int_{\partial K} \widehat{u_{\mathbf{n}}} v \; ds = 0$$

Numerical flux:

$$\widehat{u_{\mathbf{n}}} = \beta_0 \frac{[u]}{h_K} + \overline{u_{\mathbf{n}}} + \beta_1 h_K [u_{\mathbf{nn}}]$$

Solution average evolution:

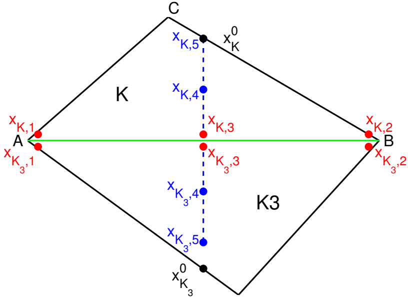
$$\overline{u}_K^{n+1} = \overline{u}_K^n + \frac{\Delta t}{area(K)} \int_{\partial K} \widehat{u_{\mathbf{n}}} \; ds$$



New idea: calculate $\int_{\partial K} \widehat{u_{\mathbf{n}}} v \; ds$ directly!

New idea for triangular mesh

$$\overline{u}_K^{n+1} = \overline{u}_K^n + \frac{\Delta t}{area(K)} \left\{ \int_{AB} \widehat{u_{\mathbf{n}}}\, ds + \int_{BC} \widehat{u_{\mathbf{n}}}\, ds + \int_{CA} \widehat{u_{\mathbf{n}}}\, ds \right\}$$



Let's use edge AB shared by elements K and K_3 to illustrate the solution points selected to calculate

$$\int_{AB} \widehat{u_{\mathbf{n}}}\, ds = \frac{\beta_0}{h_K} \int_{AB} [u]\, ds + \int_{AB} \overline{u_{\mathbf{n}}}\, ds + \beta_1 h_K \int_{AB} [u_{\mathbf{nn}}]\, ds.$$

$u_K^n(x,y)$ is a **quadratic** polynomial.

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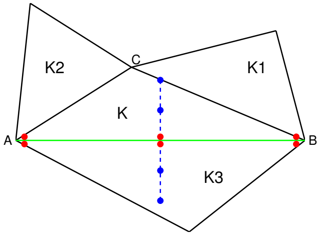
3rd order M-P-S DDG schemes



$$\overline{u}_K^{n+1} = H(u_K^n(x,y), u_{K_1}^n(x,y), u_{K_2}^n(x,y), u_{K_3}^n(x,y)) = H(\Uparrow,\Uparrow,\Uparrow,\Uparrow)$$

Theorem

Given $m \le u_K^n(x,y) \le M$ *for all* K*, we have* $m \le \overline{u}_K^{n+1} \le M$ *provided,*

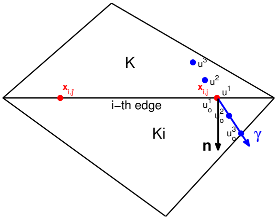
$$\beta_0 \ge \frac{9}{4} - 6\beta_1, \quad \frac{1}{8} \le \beta_1 \le \frac{1}{4}, \quad \lambda = \frac{\Delta t}{area(K)} \le C(\beta_0, \beta_1, \mathcal{T}_h).$$

Outline

Nonlinear diffusion on triangular mesh

$$u_t - \nabla \cdot (A(u)\nabla u) = 0$$

$$\overline{u}_K^{n+1} = \overline{u}_K^n + \frac{\Delta t}{area(K)} \int_{\partial K} (\widehat{A(u)\nabla u} \cdot \mathbf{n})\, ds$$



We have

$$A(u)\nabla u \cdot \mathbf{n} = \nabla u \cdot \boldsymbol{\gamma} = ||\gamma|| u_\gamma \qquad \text{with} \qquad \boldsymbol{\gamma} = A^T(u)\mathbf{n}$$

Diffusion matrix $A(u)$ is positive definite $\Longrightarrow \boldsymbol{\gamma} \cdot \mathbf{n} = \mathbf{n}^T A(u)\mathbf{n} > 0$

Numerical flux:

$$\widehat{A(u)\nabla u} \cdot \mathbf{n} = ||\gamma|| \widehat{u_\gamma} = ||\gamma|| \left\{ \beta_0 \frac{[u]}{h_K} + \overline{u_\gamma} + \beta_1 h_K [u_{\gamma\gamma}] \right\}$$

Outline

2-D M-P-S limiter

$$\widetilde{u}_K^n(x,y) = \theta(u_K^n(x,y) - \overline{u}_K^n) + \overline{u}_K^n, \quad \theta = \min\left\{1, \left|\frac{M - \overline{u}_K^n}{M_K - \overline{u}_K^n}\right|, \left|\frac{m - \overline{u}_K^n}{m_K - \overline{u}_K^n}\right|\right\},$$

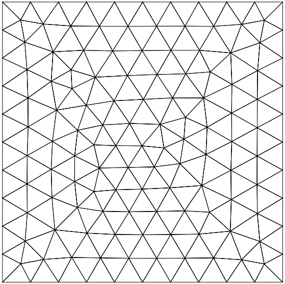
with

$$M_K = \max_{(x,y)\in K} u_K^n(x,y), \quad m_K = \min_{(x,y)\in K} u_K^n(x,y).$$

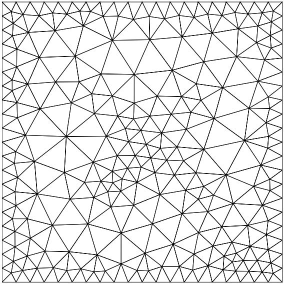
Algorithm

1. At time level t^n, apply M-P-S limiter to $u_K^n(x,y)$ and obtain $\widetilde{u}_K^n(x,y)$
2. Apply DDG scheme to $\widetilde{u}_K^n(x,y)$ and evolve in time with SSP RK method and march forward the solution to the next time level $u_K^{n+1}(x,y)$

Mesh



(a) Triangular mesh



(b) Mesh with obtuse triangles

Accuracy Tests

$$u_t - \epsilon \Delta u = 0 \qquad \text{with} \qquad \epsilon = 1$$

$$\text{exact solution: } u(x,y,t) = e^{-8\pi^2 \epsilon t} \sin(2\pi(x+y))$$

order / h	no M-P-S limiter				with M-P-S limiter			
	L^2 error	L^∞ error	$u_{\min} - ue_{\min}$	$u_{\max} - ue_{\max}$	L^2 error	L^∞ error	$u_{\min} - ue_{\min}$	$u_{\max} - ue_{\max}$
0.117			-3.91e-03	1.28e-03			0	0
0.0587	2.64	3.10	-2.40e-04	1.95e-04	2.71	3.10	0	0
0.0293	2.90	3.01	-1.49e-05	2.60e-05	2.91	3.01	0	0
0.0147	2.99	2.98	-9.61e-07	3.27e-06	2.99	2.98	0	0
0.00733	2.99	2.99	-5.82e-08	4.11e-07	2.99	2.99	0	0

Table: Accuracy table on triangular mesh 1(a) at $t = 0.0001$.

order / h	no M-P-S limiter				with M-P-S limiter			
	L^2 error	L^∞ error	$u_{\min} - ue_{\min}$	$u_{\max} - ue_{\max}$	L^2 error	L^∞ error	$u_{\min} - ue_{\min}$	$u_{\max} - ue_{\max}$
0.148			-1.17e-02	9.70e-03			0	0
0.0741	2.74	3.00	-7.86e-04	5.49e-05	3.11	2.85	0	0
0.0371	2.87	3.06	-2.66e-05	4.24e-05	2.90	3.22	0	0
0.0185	2.97	3.02	-1.12e-06	9.22e-07	2.97	3.02	0	0
0.00927	2.99	3.01	-7.17e-08	1.33e-08	2.99	3.01	0	0

Table: Accuracy table on unstructured mesh 1(b) with obtuse triangles at 0.0001.

2-D porous medium equations

$$u_t - \Delta(u^2) = 0$$

Positivity-preserving $m = 0$

2-D porous medium equations

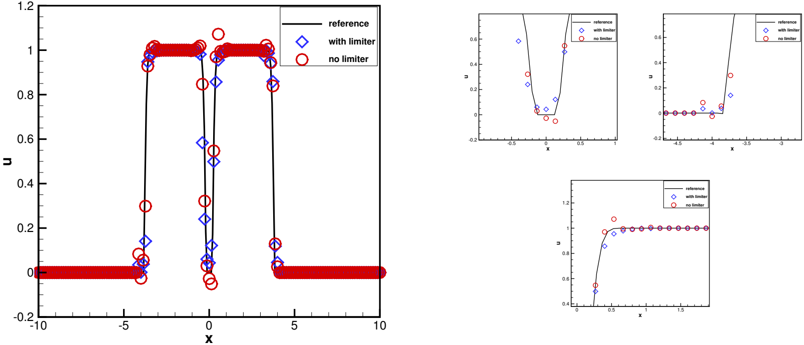


Figure: The cut of the DDG interface correction solution along line $x + y = 0$ at $t = 0.005$. Red circle symbol: no M-P-S limiter. Blue diamond symbol: M-P-S limiter applied.

Incompressible Navier-Stokes equations

vorticity stream-function formulation

$$\begin{cases} w_t + (uw)_x + (vw)_y = \frac{1}{Re}\nabla w & (1) \\ \Delta\phi = w & (2) \\ \langle u, v\rangle = \langle -\phi_y, \phi_x\rangle & (3) \end{cases}$$

We adopt Liu and Shu's method† that couple stream function ϕ with P^2 continuous finite element and DG for vorticity w.

$$\boxed{w^n} \xrightarrow[(2)]{\text{CFE}} \boxed{\phi^{n+1}} \xrightarrow[(3)]{} \boxed{u^{n+1}, v^{n+1}} \xrightarrow[SSP\ RK\ (1)]{DDG-IC} \boxed{w^{n+1}}$$

We carry out two tests: accuracy check and vortex patch problem.

† J.-G. Liu and C.-W. Shu (2000). "A high-order discontinuous Galerkin method for 2D incompressible flows". In: *J. Comput. Phys.*

Accuracy check

$$w(x,y,t) = -2\sin(x)\sin(y)e^{-\frac{2t}{Re}}$$

$h/2\pi$ \ order	no M-P-S limiter				with M-P-S limiter			
	L^2 error	L^∞ error	$u_{\min} - ue_{\min}$	$u_{\max} - ue_{\max}$	L^2 error	L^∞ error	$u_{\min} - ue_{\min}$	$u_{\max} - ue_{\max}$
0.117			-8.17e-03	7.72e-03			0	0
0.0587	2.82	3.16	-3.64e-04	6.41e-04	2.88	3.18	0	0
0.0293	2.89	2.82	-6.40e-05	-4.50e-06	2.89	2.82	0	-4.52e-06
0.0147	3.00	3.51	-1.29e-06	3.62e-06	3.00	3.51	0	0
0.00733	2.96	2.66	-1.53e-07	1.95e-07	2.96	2.66	0	0

Table: Mesh 1(a), final time $T = 0.1$

$h/2\pi$ \ order	no M-P-S limiter				with M-P-S limiter			
	L^2 error	L^∞ error	$u_{\min} - ue_{\min}$	$u_{\max} - ue_{\max}$	L^2 error	L^∞ error	$u_{\min} - ue_{\min}$	$u_{\max} - ue_{\max}$
0.148			-8.40e-04	-7.18e-05			0	-7.18e-05
0.0741	2.93	3.03	3.56e-05	-6.40e-05	2.93	3.03	4.15e-05	-6.40e-05
0.0371	2.86	2.90	3.36e-06	1.71e-06	2.86	2.90	3.36e-06	0
0.0185	2.99	3.07	1.79e-07	1.43e-07	2.99	3.07	1.79e-07	0

Table: Mesh 1(b) with obtuse triangles, final time $T = 0.1$

Vortex patch problem

$$\begin{cases} w_t + (uw)_x + (vw)_y = \frac{1}{Re}\nabla w \\ \Delta\phi = w, \qquad \langle u, v\rangle = \langle -\phi_y, \phi_x\rangle \end{cases}$$

$$w_0(x,y) = \begin{cases} -1, & (x,y) \in [\frac{\pi}{2}, \frac{3\pi}{2}] \times [\frac{\pi}{4}, \frac{3\pi}{4}], \\ 1, & (x,y) \in [\frac{\pi}{2}, \frac{3\pi}{2}] \times [\frac{5\pi}{4}, \frac{7\pi}{4}], \\ 0, & \textit{otherwise}. \end{cases}$$

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Vortex patch $Re = 100$

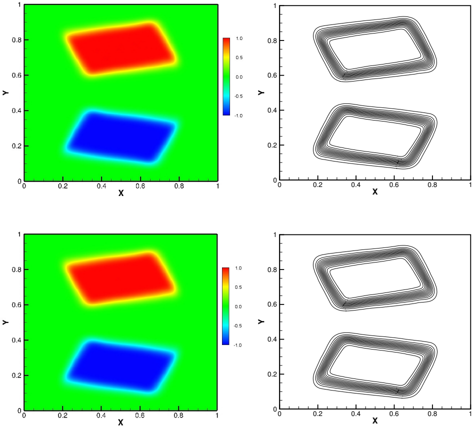

Figure: Contours of the solutions, $Re = 100$, at $t = 1$. Top: no M-P-S limiter ($w_{\min} = -1.0011$, $w_{\max} = 1.0008$); Bottom: add M-P-S limiter ($w_{\min} = -1$, $w_{\max} = 1$). 30 equally spaced contour lines are plotted.

Vortex patch $Re = 10000$

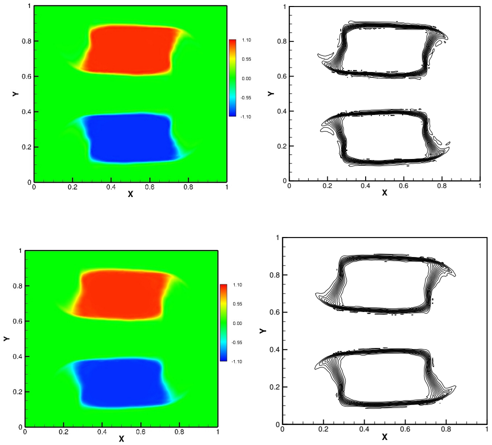

Figure: Contours of the solutions, $Re = 10000$ at $t = 5$. Top: no M-P-S limiter ($w_{\min} = -1.1211$, $w_{\max} = 1.1508$); Bottom: add M-P-S limiter ($w_{\min} = -1$, $w_{\max} = 1$). 30 equally spaced contour lines are plotted.

Vortex patch $Re = 10000$

Time evolutions of numerical vorticity w from DDG-IC schemes with M-P-S limiter, up to $t = 5$.

Vortex patch

$h/2\pi$	$Re = 100$				$Re = 10000$			
	no limiter		with limiter		no limiter		with limiter	
	$w_{\min} - we_{\min}$	$w_{\max} - we_{\max}$	$w_{\min} - we_{\min}$	$w_{\max} - we_{\max}$	$w_{\min} - we_{\min}$	$w_{\max} - we_{\max}$	$w_{\min} - we_{\min}$	$w_{\max} - we_{\max}$
0.117	-6.96e-01	6.87e-01	0	0	-8.62e-01	8.62e-01	0	0
0.0587	-2.72e-01	2.66e-01	0	0	-6.14e-01	7.82e-01	0	0
0.0293	-8.02e-02	4.22e-02	0	0	-5.14e-01	5.77e-01	0	0
0.0147	-3.17e-03	3.00e-03	0	0	-4.35e-01	4.79e-01	0	0
0.00733	-6.41e-04	7.41e-04	0	0	-3.52e-01	2.81e-01	0	0

Table: Maximum and minimum of the solutions, $Re = 100$ and $Re = 10000$ at $t = 0.1$.

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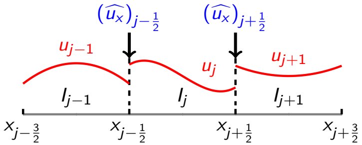
Key integredient

$$u_t - u_{xx} = 0$$

$$\int_{I_j} u_t v \, dx + \int_{I_j} u_x v_x \, dx - (\widehat{u_x} v)\Big|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} + [u](\widetilde{v_x})_{j+\frac{1}{2}} + [u](\widetilde{v_x})_{j-\frac{1}{2}} = 0$$

Euler forward in time evolution ($v = 1$)

$$\begin{aligned} \overline{u}_j^{n+1} &= \overline{u}_j^n + \frac{\Delta t}{\Delta x}\left\{(\widehat{u_x})_{j+\frac{1}{2}} - (\widehat{u_x})_{j-\frac{1}{2}}\right\} \\ &= H(u_{j-1}^n(x), u_j^n(x), u_j^{n+1}(x)) \end{aligned}$$



with $\widehat{u_x} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x} + \beta_1 \Delta x [u_{xx}]$

Degenerates to IPDG without second derivatives jump term.

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Conclusions

- Sufficient conditions are given to obtain 3rd order M-P-S DDG methods for convection diffusion problems on triangular mesh
- Second derivative jump term $[u_{\mathbf{nn}}]$ is crucial.

Future work

- Positivity-preserving for the heights of shallow water equations
- Positivity-preserving for density and pressure profiles of compressible Navier-Stokes Equations

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Thank you for your attention!

Any questions?

Motivations

- Strong stability result in the L^∞ sense
- Consistency to physical meanings (e.g. preserve positivity of density profile)

Numerical challenges

- Finite difference methods: no better than 2nd order accuracy
- Classical finite element methods (2D):
  - Linear FE: mesh without obtuse elements
  - Quadratic FE: extremely restrictive assumptions on the mesh
- Little is known about high order finite volume or spectral methods, etc.

DDG method and its variations

$$\int_{I_j} u_t v\, dx + \int_{I_j} u_x v_x\, dx - (\widehat{u_x} v)\Big|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} + \boxed{\sigma \left([u](\widetilde{v_x})_{j+\frac{1}{2}} + [u](\widetilde{v_x})_{j-\frac{1}{2}} \right)} = 0$$

- DDG: $\sigma = 0$, $\widehat{u_x}\Big|_{x_{j\pm\frac{1}{2}}} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x} + \beta_1 \Delta x [u_{xx}] + \beta_3 \Delta x^3 [u_{xxxx}] + \cdots$
- DDG with interface correction: $\sigma = 1$,

$$\begin{cases} \widehat{u_x} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x} + \beta_1 \Delta x [u_{xx}] \\ \widetilde{v_x} = \overline{v_x} \end{cases}$$

- DDG with symmetric structure: $\sigma = 1$, ($\beta_1 = 0 \rightarrow$ IPDG)

$$\begin{cases} \widehat{u_x} = \beta_{u,0} \frac{[u]}{\Delta x} + \overline{u_x} + \beta_1 \Delta x [u_{xx}] \\ \widetilde{v_x} = \beta_{v,0} \frac{[v]}{\Delta x} + \overline{v_x} + \beta_1 \Delta x [v_{xx}] \end{cases}$$

- non-symmetric DDG method: $\sigma = -1$, ($\beta_{u,0} > \beta_{v,0} \rightarrow NIPG$)

$$\begin{cases} \widehat{u_x} = \beta_{u,0} \frac{[u]}{\Delta x} + \overline{u_x} + \beta_1 \Delta x [u_{xx}] \\ \widetilde{v_x} = \beta_{v,0} \frac{[v]}{\Delta x} + \overline{v_x} + \beta_1 \Delta x [v_{xx}] \end{cases}$$

DDG method and its variations

$$\int_{I_j} u_t v \, dx + \int_{I_j} u_x \cancel{v_x} \, dx - (\widehat{u_x} v)\Big|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} + \boxed{\sigma \left([u] \cancel{(\widetilde{v_x})}_{j+\frac{1}{2}} + [u] \cancel{(\widetilde{v_x})}_{j-\frac{1}{2}} \right)} = 0$$

- DDG: $\sigma = 0$, $\widehat{u_x}\big|_{x_{j\pm\frac{1}{2}}} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x} + \beta_1 \Delta x [u_{xx}] + \beta_3 \Delta x^3 [u_{xxxx}] + \cdots$
- DDG with interface correction: $\sigma = 1$,

$$\begin{cases} \widehat{u_x} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x} + \beta_1 \Delta x [u_{xx}] \\ \widetilde{v_x} = \overline{v_x} \end{cases}$$

- DDG with symmetric structure: $\sigma = 1$, ($\beta_1 = 0 \rightarrow$ IPDG)

$$\begin{cases} \widehat{u_x} = \beta_{u,0} \frac{[u]}{\Delta x} + \overline{u_x} + \beta_1 \Delta x [u_{xx}] \\ \widetilde{v_x} = \beta_{v,0} \frac{[v]}{\Delta x} + \overline{v_x} + \beta_1 \Delta x [v_{xx}] \end{cases}$$

- non-symmetric DDG method: $\sigma = -1$, ($\beta_{u,0} > \beta_{v,0} \rightarrow NIPG$)

$$\begin{cases} \widehat{u_x} = \beta_{u,0} \frac{[u]}{\Delta x} + \overline{u_x} + \beta_1 \Delta x [u_{xx}] \\ \widetilde{v_x} = \beta_{v,0} \frac{[v]}{\Delta x} + \overline{v_x} + \beta_1 \Delta x [v_{xx}] \end{cases}$$

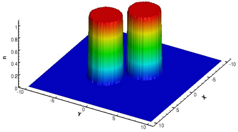
2-D porous medium equations

$$u_t - \Delta(u^2) = 0$$

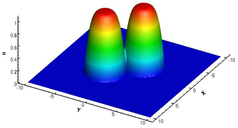
Positivity-preserving $m = 0$

2-D porous medium equations

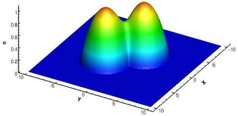
$$u_t - \Delta(u^2) = 0$$



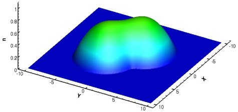
(a) t = 0



(b) t = 0.1



(c) t = 0.5



(d) t = 2

2-D porous medium equations

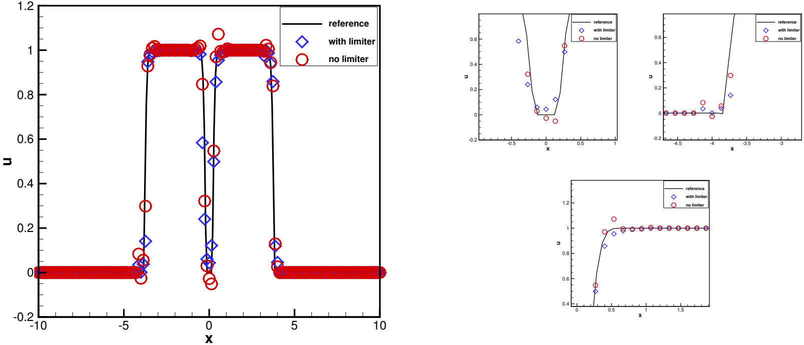


Figure: The cut of the DDG interface correction solution along line $x + y = 0$ at $t = 0.005$. Red circle symbol: no M-P-S limiter. Blue diamond symbol: M-P-S limiter applied.

2-D strongly degenerate nonlinear convection diffusion equations

$$u_t + \nabla \cdot \langle u^2, u^2 \rangle = \epsilon \nabla \cdot (\nu(u) \nabla u), \qquad \nu(u) = \begin{cases} 0, & |u| \leq 0.25, \\ 1, & |u| > 0.25. \end{cases}$$